

Lecture 5

Finished negative-definiteness proof from previous lecture.

Discussed diagonalizability of infinite-dimensional Hermitian operators. Unlike the proof of real eigenvalues, etcetera, we cannot simply repeat the proof from the matrix case (where one can proceed by induction on the dimension). In practice, however, real-symmetric operators arising from physical systems are almost always diagonalizable; the precise conditions for this lead to the "spectral theorem" of functional analysis.) (One hand-wavy argument: all physical PDEs can apparently be simulated by a sufficiently powerful computer to any desired accuracy, in principle. Since the discrete approximation is diagonalizable, and converges to the continuous solution, it would be surprising if the eigenfunctions of the continuous problem "missed" some solution. In fact, all the counter-examples of self-adjoint operators that lack a spectral theorem seem to involve unphysical solutions that oscillate infinitely fast as they approach some point, and hence cannot be captured by any discrete approximation no matter how fine.) In 18.303, we will typically just assume that all functions of interest lie in the span of the eigenfunctions, and focus on the consequences of this assumption.

Showed how this immediately tells us key properties of the solutions, if we assume the spectral theorem: Poisson's equation has a unique solution, the diffusion equation has decaying solutions (with larger eigenvalues = faster oscillations = decaying faster, making the solution smoother over time), and the wave equation has oscillating solutions.

Not only do we now understand d^2/dx^2 at a much deeper level, but you can obtain the same insights for many operators that *cannot* be solved analytically. For example, showed that the operator $d/dx [c(x) d/dx]$, which is the 1d Laplacian operator for a non-uniform "medium", is also real-symmetric positive definite if $c(x) > 0$, given the same $u(0)=u(L)=0$ boundary conditions.

As another example, considered the operator $c(x)d^2/dx^2$ for real $c(x) > 0$. This is *not* self-adjoint under the usual inner product, but *is* self-adjoint if we use the *modified* inner product $\langle u, v \rangle = \int uv/c$ with a "weight" $1/c(x)$. (This modified inner product satisfies all of our required inner-product properties for positive $c(x)$.) Therefore, $c(x)d^2/dx^2$ indeed has real, negative eigenvalues, and has eigenfunctions that are orthogonal under this new inner product. Later on, we will see more examples of how sometimes you have to change the inner product in order to understand the self-adjointness of \hat{A} .

Fortunately, it's usually pretty obvious how to change the inner product, typically some simple weighting factor that falls out of the definition of \hat{A} . (In fact, for matrices, it turns out that *every* diagonalizable matrix with real eigenvalues is Hermitian under some modified inner product. I didn't prove this, however.)

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