Lecture 8

Music and wave equations: Spent a little time relating the 18.303 theory of the vibrating string to what you hear when you listen to a stringed instrument; scales, harmonics, transposition, timbre and the Fourier series, etcetera. (See notes.) Performed a little demo on my Yamaha guitalele. To obtain a chromatic scale, each fret on the guitalele (or guitar) shortens the strings by a factor of $2^{1/12}$ (and this is why the frets get closer together as you go up the neck: they are equally spaced on a log scale).

New topic: Separation of variables: (See notes.) This is a technique to *reduce the dimensionality* of a PDE by representing the solution as a product of lower-dimensional functions. It *only works in a handful of cases*, usually as a consequence of *symmetry*, but those cases are common enough that it is important to know them. It also gives us our only analytically solvable PDE examples in more than 1d; otherwise we will have to use the computer.

Separation of Time: The most important case is the one we've already done, under another name. We solved $Au=\partial u/\partial t$ by looking for eigenfunctions $Au=\lambda u$, and then multiplying by $\exp(\lambda t)$ to get the time dependence. Similarly for $Au=\partial u^2/\partial t^2$ except with sines and cosines. In both cases, we wrote the solution as a sum of products of purely spatial functions (the eigenfunctions) and purely temporal functions like $\exp(\lambda t)$. The key point here is that we aren't assuming that the *solution* is separable, only that it can be decomposed into *linear combination* of separable functions.

Separation of Space: Here, we try to solve problems in more than one *spatial* dimension by factoring out 1d problems in one or more dimension. In particular, we will try to find *eigenfunctions* in separable form, and then write any solution as a linear combination of eigenfunctions as usual. In practice, this mainly works only in a few important cases, especially when one direction is *translationally invariant* or when the problem is *rotationally invariant*. In the former case, translational invariance in one direction (say z) allows us to write the eigenfunctions in separable form as $X(x,y)Z(z)$, where it turns out that $Z(z)=\exp(ikz)$ for some k (and X and λ will then depend on k). In the latter case, we get separable eigenfunctions $R(r)\exp(im\theta)$ where m is an integer, in 2d, and $R(r)Y_{l,m}(\theta,\varphi)$ in 3d, where $Y_{l,m}(\theta,\varphi)$ is a spherical harmonic. Also, we can *sometimes* get separable solutions for finite "box-like" domains, i.e. translationally invariant problems that have been truncated to a finite length in z.

To start with, we looked at $\nabla^2 u=\lambda u$ in a 2d $L_x\times L_y$ box with Dirichlet boundary conditions, and looked for separable solutions of the form $X(x)Y(y)$. Plugging this in and dividing by XY (the standard techniques), we get 1d eigenproblems for X and Y, and these eigenproblems ($X''=X\times$constant and $Y''=Y\times$constant) just give us our familiar sine and cosine solutions. Adding in the boundary condition, we get $\sin(n_x\pi x/L_x)\ \sin(n_y\pi x/L_y)$ eigenfunctions with eigenvalues $\lambda=-(n_x\pi/L_x)^2-(n_y\pi/L_y)^2$. As expected, these are real and negative, and the eigenfunctions are orthogonal...giving us a 2d Fourier sine series. For example, this gives us the "normal modes" of a square drum surface.

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