

Solutions to Problem Set 2 : More on the Heat Problem

18.303 Linear Partial Differential Equations

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1. Find the Fourier sine and cosine series of

$$f(x) = \frac{1}{2}(1-x), \quad 0 < x < 1.$$

- (a) State a theorem which proves convergence of each series in (a). Graph the functions to which they converge.
- (b) Show that the Fourier sine series cannot be differentiated termwise (term-by-term). Show that the Fourier cosine series converges uniformly.

Solution: The sine series is

$$\hat{f}(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

where

$$\begin{aligned} B_n &= 2 \int_0^1 f(x) \sin(n\pi x) dx = \int_0^1 (1-x) \sin(n\pi x) dx \\ &= \left[\frac{-(1-x) \cos n\pi x}{n\pi} - \frac{\sin n\pi x}{(n\pi)^2} \right]_{x=0}^1 \\ &= \frac{1}{n\pi} \end{aligned}$$

The cosine series is

$$\tilde{f}(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x)$$

where

$$A_0 = \int_0^1 f(x) dx = \frac{1}{2} \int_0^1 (1-x) dx = \frac{1}{4}$$

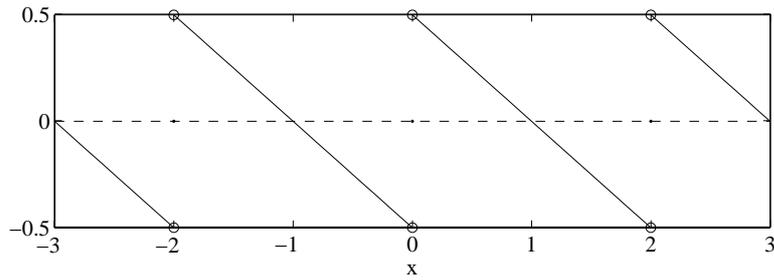


Figure 1: Sine series of $f(x)$.

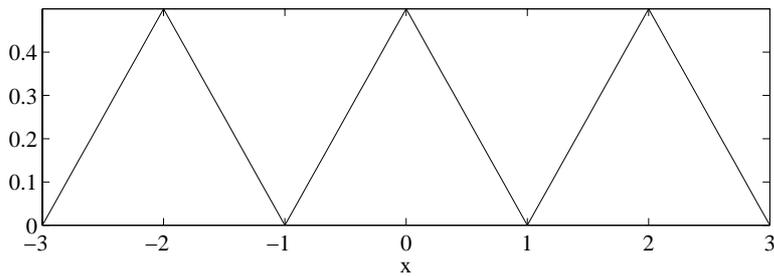


Figure 2: Cosine series of $f(x)$.

$$\begin{aligned}
 A_n &= 2 \int_0^1 f(x) \cos(n\pi x) dx = \int_0^1 (1-x) \cos(n\pi x) dx \\
 &= \left[\frac{-(1-x) \sin n\pi x}{n\pi} - \frac{\cos n\pi x}{(n\pi)^2} \right]_{x=0}^1 \\
 &= \frac{1 - \cos n\pi}{(n\pi)^2}
 \end{aligned}$$

Thus $A_{2n} = 0$ and $A_{2n-1} = 2 / ((2n-1)^2 \pi^2)$.

Both the sine and cosine series of $f(x)$ converge on the closed interval $[0, 1]$ since $f(x)$ is piecewise continuous on $0 \leq x \leq 1$ and continuous on $0 < x < 1$, as required by the theorem in the notes.

The sine series is the odd periodic extension of $f(x)$, it is even, 2-periodic and discontinuous. The sine series is plotted in Figure 1.

The cosine series is the even periodic extension of $f(x)$, it is even, 2-periodic and continuous. The cosine series is plotted in Figure 2.

Differentiating $f(x)$ gives

$$\frac{df}{dx} = -\frac{1}{2}$$

Differentiating the sine series $\hat{f}(x)$ term-by-term gives

$$\frac{d\hat{f}}{dx} = \sum_{n=1}^{\infty} \cos(n\pi x)$$

This series does not converge because the summands do not approach zero as $n \rightarrow \infty$, for any x . For a series $\sum_n a_n$ to converge, the n 'th summand a_n must approach zero as $n \rightarrow \infty$. An alternative method to show this series does not converge is to choose a single x where the series does not converge. Consider $x = 1/2$, then

$$\frac{d\hat{f}}{dx} \left(\frac{1}{2} \right) = \sum_{n=1}^{\infty} \cos \left(\frac{n\pi}{2} \right) = \sum_{m=1}^{\infty} \cos(m\pi) = \sum_{m=1}^{\infty} (-1)^m$$

where we let $m = 2n$. The partial sums

$$\sum_{m=1}^M (-1)^m = \begin{cases} 0, & M \text{ even} \\ -1, & M \text{ odd} \end{cases}$$

do not converge, and hence the series at $x = 1/2$ does not converge. In particular, the term-by-term differentiated sine series does not converge, and hence the sine series of $f(x)$, i.e. $\hat{f}(x)$ cannot be differentiated term-by-term.

[Optional] To show the cosine series $\tilde{f}(x)$ converges uniformly, we note that

$$\begin{aligned} \left| \sum_{n=1}^{\infty} A_n \cos(n\pi x) \right| &\leq \sum_{n=1}^{\infty} |A_n \cos(n\pi x)| \\ &\leq \sum_{n=1}^{\infty} |A_n| = \sum_{n=1}^{\infty} \frac{2}{(2n-1)^2 \pi^2} \leq \frac{2}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^2} \end{aligned}$$

We know $\sum_{m=1}^{\infty} \frac{1}{m^2}$ converges from our class notes, and hence the cosine series $\tilde{f}(x)$ converges uniformly. This is actually called the Boltzmann-Weierstrass M-Test.

2. Prove uniqueness for Problem 4 on Assignment 1,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \quad \frac{\partial u}{\partial x}(0, t) = 0 = u(1, t); \quad u(x, 0) = f(x)$$

where $t > 0$, $0 \leq x \leq 1$ and f is a piecewise smooth function on $[0, 1]$.

Solution: Take two solutions u_1, u_2 (twice continuously differentiable, etc).

Let

$$v(x, t) = u_1(x, t) - u_2(x, t)$$

From the PDE for u ,

$$v_t = (u_1 - u_2)_t = u_{1t} - u_{2t} = u_{1xx} - u_{2xx} = (u_1 - u_2)_{xx} = v_{xx}$$

From the BCs for u ,

$$\begin{aligned}\frac{\partial v}{\partial x}(0, t) &= \frac{\partial u_1}{\partial x}(0, t) - \frac{\partial u_2}{\partial x}(0, t) = 0 - 0 = 0 \\ v(1, t) &= u_1(1, t) - u_2(1, t) = 0 - 0 = 0\end{aligned}$$

From the IC for u ,

$$v(x, 0) = u_1(x, 0) - u_2(x, 0) = f(x) - f(x) = 0$$

To summarize, the problem for $v(x, t)$ is

$$v_t = v_{xx} \tag{1}$$

$$v_x(0, t) = 0 = v(1, t) \tag{2}$$

$$v(x, 0) = 0 \tag{3}$$

We now define the mean square of v ,

$$\bar{V}(t) = \int_0^1 v^2(x, t) dx = \int_0^1 (u_1(x, t) - u_2(x, t))^2 dx$$

Since the integrand is a non-negative function, then so must be $\bar{V}(t)$, i.e. $\bar{V}(t) \geq 0$. At $t = 0$, we use the IC (3) to obtain

$$\bar{V}(0) = \int_0^1 v^2(x, 0) dx = \int_0^1 0 dx = 0,$$

We now show $\bar{V}(t)$ is non-increasing:

$$\frac{d\bar{V}}{dt} = \int_0^1 \frac{d}{dt} (v^2(x, t)) dx = \int_0^1 2vv_t dx = \int_0^1 2vv_{xx} dx \tag{4}$$

The last step follows from the PDE (1). Integrating (4) by parts yields

$$\frac{d\bar{V}}{dt} = 2(vv_x)_{x=0}^1 - 2 \int_0^1 v_x^2 dx \tag{5}$$

From the BCs (2) on $v(x, t)$, we note that $v_x = 0$ at $x = 0$ and $v = 0$ at $x = 1$, hence $vv_x = 0$ at both $x = 0$ and 1. Hence (5) becomes

$$\frac{d\bar{V}}{dt} = -2 \int_0^1 v_x^2 dx \leq 0$$

since the integrand $(v_x)^2$ is non-negative. Thus, $\bar{V}(t)$ starts at 0 (at $t = 0$), is non-negative, and is non-increasing. Hence $\bar{V}(t) = 0$ for all $t \geq 0$. Thus

$$0 = \bar{V}(t) = \int_0^1 (v(x, t))^2 dx$$

For any given time t , since $(v(x, t))^2$ is non-negative and continuous in x , then the fact the integral is zero implies $v(x, t) = 0$ for all x . Thus

$$0 = v(x, t) = u_1(x, t) - u_2(x, t)$$

for all x , and for all time t . Thus $u_1 = u_2$ and the solution is unique.

3. Recall Problem 3 on Assignment 1,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \quad \frac{\partial u}{\partial x}(0, t) = 0 = \frac{\partial u}{\partial x}(1, t); \quad u(x, 0) = f(x)$$

where $t > 0$, $0 \leq x \leq 1$ and f is a piecewise smooth function on $[0, 1]$. Prove that the average temperature

$$\bar{u}(t) = \int_0^1 u(x, t) dx$$

is a constant for any solution of this problem. Why is this reasonable physically? Use your solution to Problem 3 (you don't have to re-derive it) to show that $\lim_{t \rightarrow \infty} u(x, t) = \bar{u}$, where \bar{u} is the constant average temperature.

Solution: To show something is constant, we want to show the derivative in time is zero. Thus, differentiating in time, we have

$$\frac{d\bar{u}}{dt} = \int_0^1 \frac{du}{dt} dx = \int_0^1 u_t dx = \int_0^1 u_{xx} dx \quad (6)$$

The last step follows from the PDE for u . Integrating (6) by parts yields

$$\frac{d\bar{u}}{dt} = (u_x)_{x=0}^1 = u_x(1, t) - u_x(0, t) = 0$$

We used the BCs for u in the last step. Thus

$$\bar{u} = \text{const} = \int_0^1 u(x, 0) dx = \int_0^1 f(x) dx$$

This is physically reasonable since in this case the rod is completely insulated, so the total energy is conserved. Thus, despite the temperature, and hence energy, being distributed in various ways along the rod, the mean temp or total energy is always the same. Recall from Problem 3 on Assignment 1 that

$$\lim_{t \rightarrow \infty} u(x, t) = A_0$$

for some constant A_0 . Thus as $t \rightarrow \infty$, the temperature becomes constant along the rod. But we just showed that the mean temp of the rod is always \bar{u} (a const). Hence

$$A_0 = \bar{u} = \int_0^1 f(x) dx$$

4. A rod of homogeneous radioactive material lies along the x -axis, $0 \leq x \leq l$. The neutron density $n(x, t)$ at position x and time t is affected by two processes - fission and diffusion. Conservation of neutrons leads to the PDE,

$$\frac{\partial n}{\partial t} = D \frac{\partial^2 n}{\partial x^2} + kn$$

where D is a diffusion coefficient and k is a fission constant, with $D > 0$, $k > 0$. Suppose that $n = 0$ at the ends of the rod. Show that the rod will explode ($n \rightarrow \infty$) if and only if

$$k > \frac{\pi^2 D}{l^2}.$$

Solution: Be careful that this problem is posed in dimensional variables. You can either non-dimensionalize it, or (easier) just work in dimensional variables. We want to find the time dependence and determine a relationship for blowup. We separate variables as

$$n(x, t) = X(x) T(t) \tag{7}$$

and substitute this into the PDE (don't forget the constant term):

$$X(x) T'(t) = DX''(x) T(t) + kX(x) T(t)$$

Dividing by $DX(x) T(t)$ yields

$$\frac{1}{D} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} + \frac{k}{D}$$

For convenience we move the k/D term to the left:

$$-\frac{k}{D} + \frac{1}{D} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

By the argument before, the left side depends only on time t , the right only on x , and hence both sides must be a constant:

$$-\frac{k}{D} + \frac{1}{D} \frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda$$

This gives two ODEs:

$$\frac{T'(t)}{T(t)} = -D\lambda + k \tag{8}$$

$$X''(x) + \lambda X(x) = 0 \tag{9}$$

From (8), we need to show $-D\lambda + k > 0$ for blowup. But first, we have to find λ by solving the ODE (9) and imposing the BCs. Introducing (9) into the BCs gives

$$0 = n(0, t) = X(0) T(t), \quad 0 = n(l, t) = X(l) T(t)$$

For non-trivial solutions, we must have $T(t)$ nonzero for some t , in which case $X(0) = X(1) = 0$. Thus the Sturm-Liouville problem for $X(x)$ is

$$X''(x) + \lambda X(x) = 0; \quad X(0) = 0 = X(1)$$

A similar argument to the one in class shows that this has eigenfunctions $X_n(x) = \sin(n\pi x/l)$ and eigenvalues $\lambda_n = n^2\pi^2/l^2$. The eigenfunctions are bounded, and will not contribute to blowup. The eigenvalues, however, control the behaviour of $T(t)$. In particular, the associated solutions for $T(t)$ are given by (8),

$$\frac{T'(t)}{T(t)} = -\frac{D}{l^2}n^2\pi^2 + k$$

For blowup, we must have

$$-n^2\frac{D\pi^2}{l^2} + k > 0$$

for some n . In other words, blowup happens if and only if

$$k > n^2\frac{D\pi^2}{l^2}$$

for some n . But $D\pi^2 < Dn^2\pi^2$ for $n > 1$, so blowup happens if and only if

$$k > \frac{D\pi^2}{l^2}$$

5. Consider the inhomogeneous generalized heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial x} + cu + g(x, t) \quad (10)$$

where b, c are constants.

(a) Show that if u is a solution to (10), then

$$v(x, t) = e^{\alpha x + \beta t}u(x, t) \quad (11)$$

satisfies the standard heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + h(x, t)$$

for suitable choices of the constants α, β and function $h(x, t)$. In this way, more complicated heat problems can be simplified.

Solution: Re-writing (11) for $u(x, t)$ gives

$$u(x, t) = e^{-\alpha x - \beta t}v(x, t) \quad (12)$$

Note that

$$\begin{aligned} u_t &= e^{-\alpha x - \beta t} (-\beta v + v_t) \\ u_x &= e^{-\alpha x - \beta t} (-\alpha v + v_x) \\ u_{xx} &= e^{-\alpha x - \beta t} (\alpha^2 v - 2\alpha v_x + v_{xx}) \end{aligned} \quad (13)$$

Substituting into the generalized heat equation (10) gives

$$v_t = v_{xx} + (b - 2\alpha) v_x + (\alpha^2 + \beta - \alpha b + c) v + g e^{\alpha x + \beta t} \quad (14)$$

To get rid of the v_x and v terms, we choose

$$\begin{aligned} b - 2\alpha &= 0 \\ \alpha^2 + \beta - \alpha b + c &= 0 \end{aligned}$$

Solving for α, β gives

$$\begin{aligned} \alpha &= b/2 \\ \beta &= -(\alpha^2 - \alpha b + c) = \frac{b^2}{4} - c \end{aligned}$$

Choosing

$$h(x, t) = g(x, t) e^{\alpha x + \beta t} = g(x, t) e^{\alpha x + \beta t}$$

and substituting for α and β in the PDE (14) gives the standard heat equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + h(x, t).$$

- (b) Now assume $b = c = 0$ and $g = g_0$ is a constant. Suppose the BCs and IC are all homogeneous,

$$u(0, t) = 0 = u(1, t); \quad u(x, 0) = 0.$$

Find the equilibrium solution $u_E(x)$ to (10) and, without using your results in part (a), transform (10) to a standard homogeneous problem for a temperature function $w(x, t)$.

Solution: The original PDE (10) becomes

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g_0$$

The equilibrium solution $u(x, t) = u_E(x)$ satisfies

$$0 = u_E'' + g_0$$

Solving for u_E gives

$$u_E = -g_0 \frac{x^2}{2} + Ax + B$$

The BCs on u_E are

$$u_E(0) = 0 = u_E(1)$$

Imposing the BCs on our general solution and solving for A, B yields

$$\begin{aligned} 0 &= u_E(0) = B \\ 0 &= u_E(1) = -g_0 \frac{1}{2} + A \end{aligned}$$

Thus $A = g_0/2$ and $B = 0$, and

$$u_E(x) = \frac{g_0}{2}x(1-x)$$

We now transform our PDE. Let

$$w(x, t) = u(x, t) - u_E(x)$$

Note that

$$\begin{aligned} w_t &= u_t \\ w_{xx} &= u_{xx} - u_E'' = u_{xx} + g_0 \end{aligned}$$

Thus,

$$w_t - w_{xx} = u_t - u_{xx} - g_0 = 0$$

The BCs for w are

$$\begin{aligned} w(0, t) &= u(0, t) - u_E(0) = 0 \\ w(1, t) &= u(1, t) - u_E(1) = 0 \end{aligned}$$

Thus $w(x, t)$ satisfies the basic heat problem,

$$\begin{aligned} w_t &= w_{xx}, & 0 < x < 1, & \quad t > 0 \\ w(0, t) &= 0 = w(1, t), & & \quad t > 0 \\ w(x, 0) &= u(x, 0) - u_E(x) = -\frac{g_0}{2}x(1-x), & & \quad 0 < x < 1. \end{aligned}$$

(c) Continuing from part (b), show that for large t ,

$$u(x, t) \approx u_E(x) + Ce^{-\pi^2 t} \sin \pi x$$

where C is some constant. Find C and comment on the physical significance of its sign. Illustrate the solution qualitatively by sketching typical

spatial temperature profiles with $t = \text{constant}$ and the temperature time profile at $x = 1/2$.

Solution: The solution for $w(x, t)$ is, from class,

$$w(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

and is well approximated by the first term, for $t \geq 1/\pi^2$,

$$w(x, t) \approx B_1 \sin(\pi x) e^{-\pi^2 t}$$

where

$$B_1 = 2 \int_0^1 \left(-\frac{g_0}{2} x(1-x) \right) \sin(\pi x) dx = -\frac{4}{\pi^3} g_0$$

Thus for $t \geq 1/\pi^2$,

$$u(x, t) = u_E(x) + w(x, t) \approx \frac{1}{2} g_0 x(1-x) - \frac{4}{\pi^3} g_0 e^{-\pi^2 t} \sin \pi x$$

So $C = -4g_0/\pi^3$. The sign of C is opposite that of g_0 , since the transient term is changing the opposite way the rod is. If $g_0 > 0$, the heat source increases the rod temp over time, so the transient term, whose magnitude gets smaller as time increases, should be negative.

[Optional] In general,

$$\begin{aligned} B_n &= 2 \int_0^1 \left(-\frac{g_0}{2} x(1-x) \right) \sin(n\pi x) dx \\ &= \frac{g_0}{n^3 \pi^3} (2 \cos \pi n + \pi n \sin \pi n - 2) \\ &= \frac{2g_0}{n^3 \pi^3} ((-1)^n - 1) \end{aligned}$$

Thus $B_{2n} = 0$ and

$$B_{2n-1} = -\frac{4g_0}{(2n-1)^3 \pi^3}$$

At $x = 1/2$, we have

$$\begin{aligned} u\left(\frac{1}{2}, t\right) &= u_E\left(\frac{1}{2}\right) + w\left(\frac{1}{2}, t\right) \\ &= \frac{1}{2} g_0 \frac{1}{2} \left(1 - \frac{1}{2}\right) + \sum_{n=1}^{\infty} B_{2n-1} \sin\left((2n-1) \frac{\pi}{2}\right) e^{-(2n-1)^2 \pi^2 t} \\ &= \frac{g_0}{8} - \frac{4g_0}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^3} e^{-(2n-1)^2 \pi^2 t} \end{aligned}$$

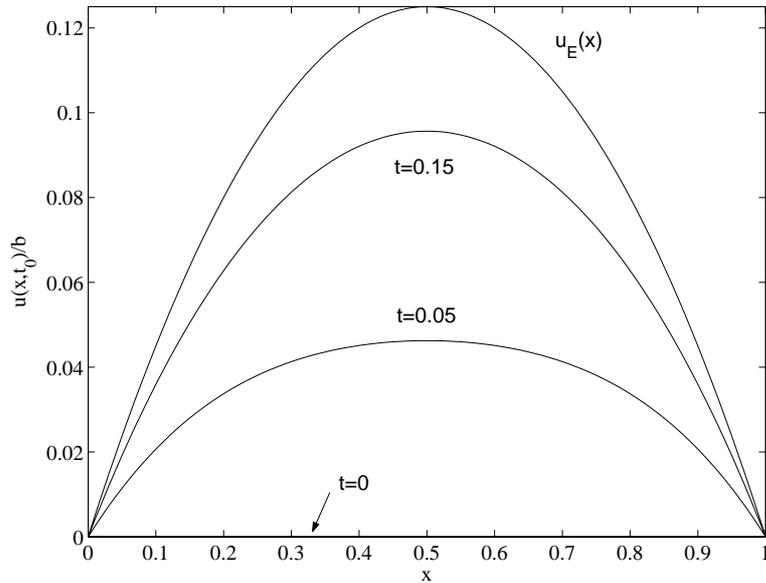


Figure 3: Plots of $u(x, t_0)$ for $t_0 = 0, 0.05, 0.15$ and of $u_E(x)$.

For plots, note that $u_E(x)$ is an upside-down parabola whose vertex is at $(1/2, -g_0/8)$ in the ux -plane. Also,

$$u\left(\frac{1}{2}, t\right) \approx u_E\left(\frac{1}{2}\right) - \frac{4g_0}{\pi^3} e^{-\pi^2 t} \sin\left(\frac{\pi}{2}\right) = \frac{g_0}{8} - \frac{4g_0}{\pi^3} e^{-\pi^2 t}.$$

The plots are given in Figures 3 and 4.

6. Consider the inhomogeneous heat problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}; \quad u(0, t) = a(t), \quad u(1, t) = b(t); \quad u(x, 0) = f(x) \quad (15)$$

with inhomogeneous boundary conditions, where $a(t)$ and $b(t)$ are given continuous functions of time.

(a) Show that (15) has at most one solution.

Solution: The uniqueness proof is very similar to that in Problem 2. Take two solutions u_1, u_2 and let

$$v(x, t) = u_1(x, t) - u_2(x, t)$$

As before, from the PDE for u , we have

$$v_t = v_{xx}$$

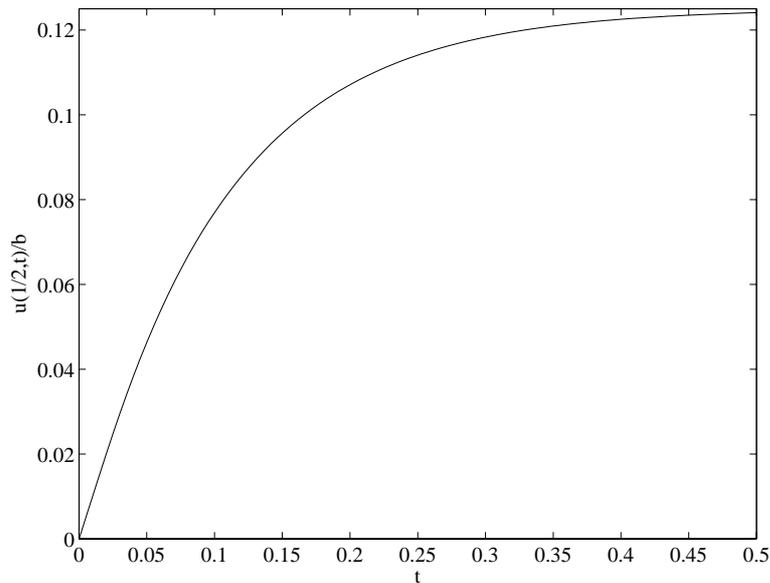


Figure 4: Plot of $u(1/2, t)$.

From the BCs for u ,

$$\begin{aligned} v(0, t) &= u_1(0, t) - u_2(0, t) = a(t) - a(t) = 0 \\ v(1, t) &= u_1(1, t) - u_2(1, t) = b(t) - b(t) = 0 \end{aligned}$$

From the IC for u ,

$$v(x, 0) = u_1(x, 0) - u_2(x, 0) = f(x) - f(x) = 0$$

To summarize, the problem for $v(x, t)$ is

$$\begin{aligned} v_t &= v_{xx} \\ v(0, t) &= 0 = v(1, t) \\ v(x, 0) &= 0 \end{aligned}$$

We considered this case in class (I expect you to show the steps), and showed $v(x, t) = 0 = u_1(x, t) - u_2(x, t)$. Thus $u_1(x, t) = u_2(x, t)$ and the solution is unique.

- (b) Transform (15) into a standard problem (i.e. one with homogeneous BCs) in terms of the unknown function $v(x, t)$.

Solution: There are several ways this can be done. We have a problem involving a homogeneous PDE and inhomogeneous BCs. We want to change this to a problem with an inhomogeneous PDE and homogeneous BCs.

The reason you might want to do this is that many over-the-counter heat equation solvers solve PDEs like $u_t = u_{xx} + h(x, t)$ (the standard Heat Problem), but assume zero BCs. We need to subtract a function from $u(x, t)$ that equals $a(t)$ at $x = 0$ and $b(t)$ at $x = 1$. For each t , one such function is simply a line (in x) from $a(t)$ to $b(t)$,

$$l(x, t) = a(t) + (b(t) - a(t))x$$

Let

$$w(x, t) = u(x, t) - l(x, t) = u(x, t) - (a(t) + (b(t) - a(t))x)$$

Notice that the BCs for $w(x, t)$ are now homogeneous,

$$\begin{aligned} w(0, t) &= u(0, t) - a(t) = 0 \\ w(1, t) &= u(1, t) - b(t) = 0 \end{aligned}$$

Since what we subtracted from $u(x, t)$ is linear in x , it disappears in w_{xx} (check for yourself):

$$w_{xx} = u_{xx} - l_{xx} = u_{xx} \tag{16}$$

Also,

$$\begin{aligned} w_t &= u_t - l_t \\ &= u_t - a'(t) - (b'(t) - a'(t))x \\ &= u_{xx} - a'(t) - (b'(t) - a'(t))x \\ &= w_{xx} - a'(t) - (b'(t) - a'(t))x \end{aligned}$$

where we used the PDE for u to get the second last step, and (16) to obtain the last step. Thus the PDE for $w(x, t)$ is now inhomogeneous,

$$w_t = w_{xx} + h(x, t)$$

where

$$h(x, t) = -a'(t) - (b'(t) - a'(t))x$$

- (c) Now assume $a(t)$, $b(t)$ are constants and $f(x) = 0$. Find the equilibrium solution $u_E(x)$ to (15).

Solution: If $a(t) = a$ and $b(t) = b$ are constants, we go back to the original problem (15) for $u(x, t)$ and find the equilibrium solution $u(x, t) = u_E(x)$:

$$u_E'' = 0; \quad u_E(0) = a, \quad u_E(1) = b$$

Integrating twice and imposing the BCs gives

$$u_E(x) = a + (b - a)x$$

(d) Continuing from part (c), show that for large t ,

$$u(x, t) \approx u_E(x) + Ce^{-\pi^2 t} \sin \pi x$$

where C is some constant. Find C . Hint: use the approximate solution for the homogeneous heat problem we considered in class.

Solution: As in 5(c), we let $w(x, t) = u(x, t) - u_E(x)$ and obtain

$$w_{xx} = u_{xx} - u_E'' = u_{xx} = u_t = w_t$$

Thus,

$$w_t = w_{xx}$$

The BCs for w are

$$w(0, t) = w(0, t) - u_E(0) = 0$$

$$w(1, t) = w(1, t) - u_E(1) = 0$$

Thus $w(x, t)$ satisfies the basic heat problem,

$$\begin{aligned} w_t &= w_{xx}, & 0 < x < 1, & & t > 0 \\ w(0, t) &= 0 = w(1, t), & & & t > 0 \\ w(x, 0) &= u(x, 0) - u_E(x) = -(a + (b - a)x), & & & 0 < x < 1. \end{aligned}$$

From class, the solution is

$$w(x, t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

and is well approximated by the first term, for $t \geq 1/\pi^2$,

$$w(x, t) \approx B_1 \sin(\pi x) e^{-\pi^2 t}$$

where

$$B_1 = -2 \int_0^1 (a + (b - a)x) \sin(\pi x) dx = -\frac{2}{\pi} (a + b)$$

Thus for $t \geq 1/\pi^2$,

$$u(x, t) = u_E(x) + w(x, t) \approx a + (b - a)x - \frac{2}{\pi} (a + b) e^{-\pi^2 t} \sin \pi x$$

So $C = -2(a + b)/\pi$.

7. Fourier's Ring. Consider a slender homogeneous ring which is insulated laterally. Let x denote the distance along the ring and let l be the circumference of the ring. From physics (see Haberman §2.4.2), the temperature $u(x, t)$ satisfies, in dimensionless form,

$$\begin{aligned} u_t &= u_{xx}; & 0 < x < 2, & \quad t > 0 & \quad (17) \\ u(x+2, t) &= u(x, t); & t > 0 \\ u(x, 0) &= f(x) & 0 < x < 2. \end{aligned}$$

The boundary condition (middle equation) merely states that the temperature is continuous as you go around the ring.

(a) Use separation of variables and Fourier Series to obtain the solution to (17):

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} (A_n \cos(n\pi x) + B_n \sin(n\pi x))$$

Give formulae for the coefficients A_n, B_n in terms of $f(x)$.

Solution: As before, we separate variables $u(x, t) = X(x)T(t)$ in the PDE to obtain

$$\frac{T'}{T} = \frac{X''}{X}$$

Since the left side depends only on t and the right on x , both must equal a constant:

$$\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

The BCs imply

$$X(x+2) = X(x)$$

The problem for $X(x)$ is

$$X'' + \lambda X = 0; \quad X(x+2) = X(x)$$

We first try $\lambda = 0$. This implies $X(x) = A + Bx$, and the periodicity condition $X(x+2) = X(x)$ implies $B = 0$. Thus $\lambda = 0$ and $X_0(x) = A_0$ works.

For $\lambda < 0$, we have

$$X(x) = c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x}$$

and periodicity implies

$$c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x} = c_1 e^{\sqrt{-\lambda}(x+2)} + c_2 e^{-\sqrt{-\lambda}(x+2)}$$

Rearranging yields

$$c_1 \left(1 - e^{2\sqrt{-\lambda}}\right) e^{2\sqrt{-\lambda}x} = -c_2 \left(1 - e^{-2\sqrt{-\lambda}}\right)$$

The right hand side is a constant. However, since $\lambda < 0$, then as x changes, so does $e^{2\sqrt{-\lambda}x}$. Thus the left hand side will change - unless the coefficients are both zero! Thus,

$$c_1 \left(1 - e^{2\sqrt{-\lambda}}\right) = 0 = c_2 \left(1 - e^{-2\sqrt{-\lambda}}\right)$$

But since $\lambda < 0$, the terms $1 - e^{\pm 2\sqrt{-\lambda}}$ are not zero, and hence $c_1 = c_2 = 0$ (the trivial solution). Thus we discard $\lambda < 0$.

The last case is $\lambda > 0$:

$$X(x) = c_1 e^{i\sqrt{\lambda}x} + c_2 e^{-i\sqrt{\lambda}x}$$

and, similarly to the case $\lambda < 0$, periodicity implies

$$c_1 \left(1 - e^{2i\sqrt{\lambda}}\right) e^{2i\sqrt{\lambda}x} = -c_2 \left(1 - e^{-2i\sqrt{\lambda}}\right)$$

Since $e^{2i\sqrt{\lambda}x}$ changes as x varies, and the right side is constant, then the coefficients must be zero. To have a non-trivial solution, c_1 and c_2 cannot both be zero, and hence either

$$1 - e^{2i\sqrt{\lambda}} = 0 \quad \text{or} \quad 1 - e^{-2i\sqrt{\lambda}} = 0$$

Note that the only way $e^{2i\sqrt{\lambda}}$ (or $e^{-2i\sqrt{\lambda}}$) can be 1 is if $2\sqrt{\lambda}$ is a multiple of 2π . Thus

$$2\sqrt{\lambda} = 2n\pi$$

Solving for λ gives the eigenvalues

$$\lambda_n = n^2\pi^2$$

Note: there are several ways to find λ_n - you could use the "useful result" from class (we've actually just gone through an alternate proof of it), or imposed $X(0) = X(2)$ and $X'(0) = X'(2)$ and solved two algebraic equations. We could have also combined the cases $\lambda < 0$ and $\lambda > 0$, but I wanted to separate them for clarity. Use the method you find simplest.

We can now write $X(x)$ in terms of real valued functions, using the rule

$$e^{i\sqrt{\lambda}x} = \cos \sqrt{\lambda}x + i \sin \sqrt{\lambda}x = \cos n\pi x + i \sin \sqrt{\lambda}x,$$

we have

$$X_n(x) = A_n \cos(n\pi x) + B_n \sin(n\pi x)$$

for some constants A_n, B_n . The solutions for $T(t)$ are $T_n(t) = e^{-n^2\pi^2 t}$. Summing over all n (don't forget $u_0(x, t) = A_0 e^0 = A_0$) gives

$$u(x, t) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\pi x) + B_n \sin(n\pi x)) e^{-n^2\pi^2 t} \quad (18)$$

The A_n 's and B_n 's are found by imposing the IC.

At $t = 0$, we impose the IC to obtain

$$f(x) = u(x, 0) = A_0 + \sum_{n=1}^{\infty} (A_n \cos(n\pi x) + B_n \sin(n\pi x)) \quad (19)$$

We now need to derive orthogonality conditions for $\cos(n\pi x)$ and $\sin(n\pi x)$:

$$\begin{aligned} & \int_0^2 \cos(n\pi x) \cos(m\pi x) dx \\ &= \frac{1}{2} \int_0^2 \cos((m-n)\pi x) dx + \frac{1}{2} \int_0^2 \cos((m+n)\pi x) dx \\ &= \frac{1}{2} \int_0^2 \cos((m-n)\pi x) dx \\ &= \delta_{mn} \end{aligned}$$

where $\delta_{mn} = 1$ if $m = n$ and 0 if $m \neq n$ is the Kronecker delta. Similarly,

$$\begin{aligned} & \int_0^2 \cos(n\pi x) \sin(m\pi x) dx = 0 \\ & \int_0^2 \sin(n\pi x) \sin(m\pi x) dx = \delta_{mn} \end{aligned}$$

Integrating (19) from $x = 0$ to 2 gives

$$\begin{aligned} \int_0^2 f(x) dx &= A_0 \int_0^2 dx + \sum_{n=1}^{\infty} \left(A_n \int_0^2 \cos(n\pi x) dx + B_n \int_0^2 \sin(n\pi x) dx \right) \\ &= 2A_0 + \sum_{n=1}^{\infty} (A_n 0 + B_n 0) = 2A_0 \end{aligned}$$

Multiplying (19) by $\cos(m\pi x)$ and integrating from $x = 0$ to 2 yields

$$\begin{aligned}
 & \int_0^2 f(x) \cos(m\pi x) dx \\
 = & A_0 \int_0^2 \cos(m\pi x) dx \\
 & + \sum_{n=1}^{\infty} \left(A_n \int_0^2 \cos(n\pi x) \cos(m\pi x) dx + B_n \int_0^2 \sin(n\pi x) \cos(m\pi x) dx \right) \\
 = & 0 + \sum_{n=1}^{\infty} (A_n \delta_{mn} + B_n 0) \\
 = & A_m
 \end{aligned}$$

Multiplying (19) by $\sin(m\pi x)$ and integrating from $x = 0$ to 2 yields

$$\begin{aligned}
 & \int_0^2 f(x) \sin(m\pi x) dx \\
 = & A_0 \int_0^2 \sin(m\pi x) dx \\
 & + \sum_{n=1}^{\infty} \left(A_n \int_0^2 \cos(n\pi x) \sin(m\pi x) dx + B_n \int_0^2 \sin(n\pi x) \sin(m\pi x) dx \right) \\
 = & 0 + \sum_{n=1}^{\infty} (A_n 0 + B_n \delta_{mn}) \\
 = & B_m
 \end{aligned}$$

To summarize, the Fourier coefficients are given by

$$A_0 = \frac{1}{2} \int_0^2 f(x) dx = \text{avg of } f(x) \text{ over } [0, 2]$$

and for $m > 0$,

$$A_m = \int_0^2 f(x) \cos(m\pi x) dx, \quad B_m = \int_0^2 f(x) \sin(m\pi x) dx.$$

(b) Prove that (17) has at most one solution.

Solution: The uniqueness proof is similar to that in Problem 2. Take two solutions u_1, u_2 and let

$$v(x, t) = u_1(x, t) - u_2(x, t)$$

As before, from the PDE for u , we have

$$v_t = v_{xx}$$

From the BCs for u ,

$$v(x+2, t) = u_1(x+2, t) - u_2(x+2, t) = u_1(x, t) - u_2(x, t) = v(x, t)$$

From the IC for u ,

$$v(x, 0) = u_1(x, 0) - u_2(x, 0) = f(x) - f(x) = 0$$

To summarize, the problem for $v(x, t)$ is

$$\begin{aligned} v_t &= v_{xx} \\ v(x+2, t) &= v(x, t) \\ v(x, 0) &= 0 \end{aligned}$$

This problem is the same as the one for $u(x, t)$ except $f(x) = 0$. You may be tempted to plug $f(x) = 0$ into the equations above for A_n and B_n and then claim $v(x, t) = 0$. The problem is that you can't use the solution we found using separation of variables, because we are trying to prove there aren't other solution methods that give different answers. So we proceed as in class and problem 2. Define a function

$$\Delta(t) = \int_0^2 v^2(x, t) dx = \int_0^2 (u_1(x, t) - u_2(x, t))^2 dx$$

Note that

$$\Delta(0) = \int_0^2 v^2(x, 0) dx = 0$$

and since the integrand is non-negative,

$$\Delta(t) \geq 0.$$

As usual, we now show that $\Delta(t)$ is non-increasing,

$$\frac{d\Delta}{dt} = \int_0^2 2vv_t dx = \int_0^2 2vv_{xx} dx = 2(vv_x)_{x=0}^2 - 2 \int_0^2 v_x^2 dx$$

where we used the PDE for the 2nd step and integration by parts for the 3rd step. Note that since v and v_x are periodic (since $v(x+2, t) = v(x, t)$ holds for all x we can differentiate it in x to obtain $v_x(x+2, t) = v_x(x, t)$), hence

$$\begin{aligned} (vv_x)_{x=0}^2 &= v(2, t)v_x(2, t) - v(0, t)v_x(0, t) \\ &= v(0, t)v_x(0, t) - v(0, t)v_x(0, t) \\ &= 0 \end{aligned}$$

Thus

$$\frac{d\Delta}{dt} = -2 \int_0^2 v_x^2 dx \leq 0$$

Thus $\Delta(t)$ is non-negative, starts at zero, and is non-increasing. Thus $\Delta(t) = 0$ for all time, and since the integrand $v^2(x, t)$ is non-negative and continuous in x , then $v(x, t) = 0$ for all x , and any time t . Hence $u_1(x, t) = u_2(x, t)$ for all x and t .

8. Determine which of the following operators are linear:

Take two continuously differentiable functions v and w , and a real number k (a constant).

(a) $L(u) = u_t + x^2 u_{xx}$

Solution: Check addition rule:

$$\begin{aligned} L(v+w) &= (v+w)_t + x^2 (v+w)_{xx} \\ &= v_t + w_t + x^2 (v_{xx} + w_{xx}) \\ &= v_t + x^2 v_{xx} + w_t + x^2 w_{xx} \\ &= L(v) + L(w) \end{aligned}$$

Thus L satisfies the addition rule. Check scalar multiplication:

$$\begin{aligned} L(kv) &= (kv)_t + x^2 (kv)_{xx} \\ &= kv_t + x^2 kv_{xx} \\ &= k(v_t + x^2 v_{xx}) \\ &= kL(v) \end{aligned}$$

Thus L satisfies the scalar multiplication rule. Thus L is a linear operator.

(b) $L(u) = uu_{xx}$

Solution: Check scalar multiplication:

$$L(kv) = (kv)(kv)_{xx} = k^2 v v_{xx} = k^2 L(v)$$

Thus L does NOT satisfy the scalar multiplication rule and is not linear!

(c) $L(u) = e^{x^2 t} u_{xx}$

Solution: Check addition rule:

$$\begin{aligned} L(v+w) &= e^{x^2 t} (v+w)_{xx} \\ &= e^{x^2 t} (v_{xx} + w_{xx}) \\ &= e^{x^2 t} v_{xx} + e^{x^2 t} w_{xx} \\ &= L(v) + L(w) \end{aligned}$$

Thus L satisfies the addition rule. Check scalar multiplication:

$$L(kv) = e^{x^2t}(kv)_{xx} = ke^{x^2t}v_{xx} = kL(v)$$

Thus L satisfies the scalar multiplication rule. Thus L is a linear operator.

(d) $L(u) = u_{xx} - \int_0^1 u_t(y, t) dy$

Solution: Check addition rule:

$$\begin{aligned} L(v+w) &= (v+w)_{xx} - \int_0^1 (v(y,t) + w(y,t))_t dy \\ &= v_{xx} + w_{xx} - \int_0^1 (v_t(y,t) + w_t(y,t)) dy \\ &= v_{xx} + w_{xx} - \int_0^1 v_t(y,t) dy - \int_0^1 w_t(y,t) dy \\ &= v_{xx} - \int_0^1 v_t(y,t) dy + w_{xx} - \int_0^1 w_t(y,t) dy \\ &= L(v) + L(w) \end{aligned}$$

Thus L satisfies the addition rule. Check scalar multiplication:

$$\begin{aligned} L(kv) &= (kv)_{xx} - \int_0^1 (kv(y,t))_t dy \\ &= kv_{xx} - k \int_0^1 v_t(y,t) dy \\ &= k \left(v_{xx} - \int_0^1 v_t(y,t) dy \right) = kL(v) \end{aligned}$$

Thus L satisfies the scalar multiplication rule. Thus L is a linear operator.