Solutions to Problems for Infinite Spatial Domains and the Fourier Transform

18.303 Linear Partial Differential Equations

Matthew J. Hancock

Fall 2006

1 Problem 1

Do problem 10.4.3 in Haberman (p 469). The answer for (a) is in the back - please show how to get that answer. After doing parts (a), (b), solve the same PDE on the semi-infinite rod $\{x \geq 0\}$ with an insulated BC at $x = 0$:

$$\frac{\partial u}{\partial x} = 0 \qquad \text{at} \qquad x = 0$$

and the IC

$$u(x, 0) = \delta(x - 1), \qquad x > 0.$$

We also assume u is bounded as $x \to \infty$.

Solutions: (a) The problem 10.4.3 is to solve the diffusion equation with convection,

$$\begin{aligned} \frac{\partial u}{\partial t} &= k\frac{\partial^2 u}{\partial x^2} + c\frac{\partial u}{\partial x}, \qquad -\infty < x < \infty, \qquad t > 0, \\ u(x, 0) &= f(x), \qquad -\infty < x < \infty. \end{aligned}$$

Define the Fourier Transform as

$$\mathcal{F}[u(x, t)] = \bar{U}(\omega, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x, t)\, e^{i\omega x} dx$$

Taking the Fourier Transform of the PDE gives, from our rules in class,

$$\frac{\partial}{\partial t}\bar{U}(\omega, t) = -k\omega^2\bar{U}(\omega, t) - ci\omega\bar{U}(\omega, t) = \left(-k\omega^2 - ci\omega\right)\bar{U}(\omega, t)$$

Integrating gives

$$\bar{U}(\omega, t) = C(\omega)\, e^{-k\omega^2 t - ci\omega t}$$

Imposing the IC gives

$$C(\omega) = \bar{U}(\omega, 0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} u(x,0)\, e^{i\omega x} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{i\omega x} dx.$$

Thus $C(\omega) = F(\omega)$ is the Fourier Transform of $f(x)$. Lastly,

$$\bar{U}(\omega, t) = F(\omega)\, e^{-k\omega^2 t} e^{-ci\omega t}$$

Note the inverse FT's:

$$\mathcal{F}^{-1}[F(\omega)] = f(x), \qquad \mathcal{F}^{-1}\left[e^{-k\omega^2 t}\right] = \sqrt{\frac{\pi}{kt}} e^{-x^2/4kt}$$

To find the inverse FT, we use the convolution theorem to obtain, as in class,

$$\mathcal{F}^{-1}\left[F(\omega)\, e^{-k\omega^2 t}\right] = \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-s)^2}{4kt}\right) ds$$

We now use the Shifting Theorem (Table on p 468),

$$\begin{aligned}
\mathcal{F}^{-1}\left[e^{-i\omega\beta} G(\omega)\right] &= \int_{-\infty}^{\infty} e^{-i\omega\beta} G(\omega)\, e^{-i\omega x} d\omega \\
&= \int_{-\infty}^{\infty} G(\omega)\, e^{-i\omega(\beta+x)} d\omega \\
&= g(x+\beta)
\end{aligned}$$

so that

$$\begin{aligned}
u(x,t) &= \mathcal{F}^{-1}\left[\bar{U}(\omega, t)\right] \\
&= \mathcal{F}^{-1}\left[e^{-ci\omega t} F(\omega)\, e^{-k\omega^2 t}\right] \\
&= \int_{-\infty}^{\infty} \frac{f(s)}{\sqrt{4\pi kt}} \exp\left(-\frac{(x+ct-s)^2}{4kt}\right) ds
\end{aligned}$$

(b) Consider the IC $f(x) = \delta(x)$. Substituting $f(s) = \delta(s)$ gives

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\delta(s)}{\sqrt{4\pi kt}} \exp\left(-\frac{(x+ct-s)^2}{4kt}\right) ds$$

To evaluate the integrals, we use the sifting property of the δ function:

$$\int_a^b \delta(s-c)\, g(s) = g(c)$$

for $a < c < b$. Thus

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x+ct)^2}{4kt}\right)$$

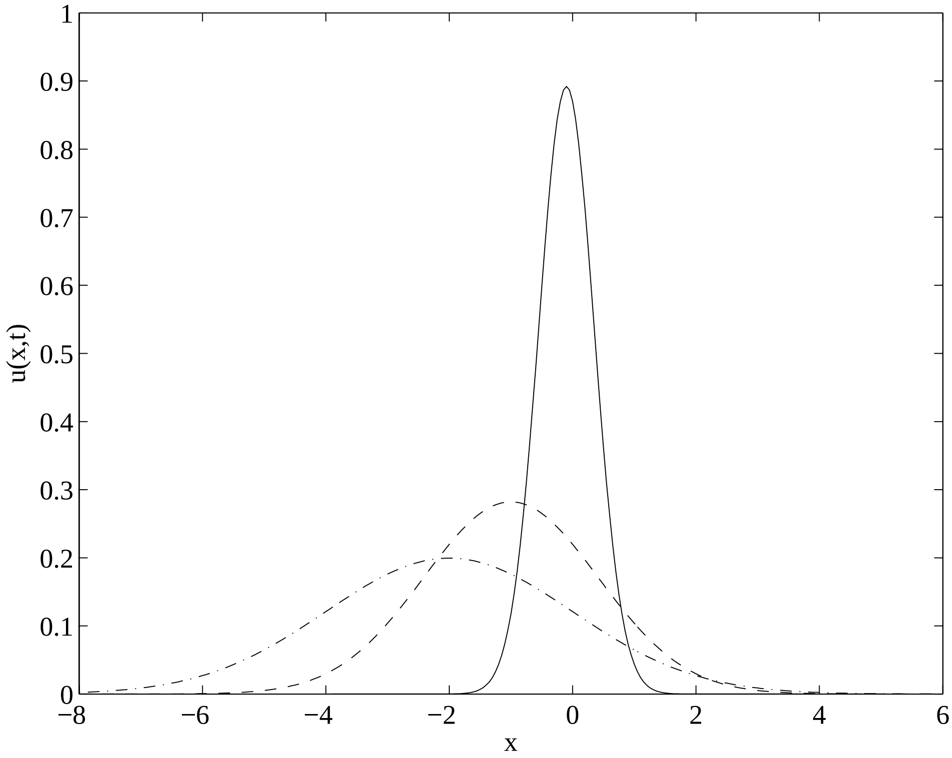

Figure 1: Sketch of $u(x,t)$ with $c = k$, for $kt = 0.1$ (solid), 1 (dashed) and 2 (dash-dot).

Plots are given in Figure 1. The convective term cu_x moves the peak to the left, as the lump becomes more spread out (diffuse) due to the diffusion term ku_{xx}.

(c) For the semi-infinite rod, things are different (e.g. see problem 10.5.14). First, we use the methods of PSet 2, Q5a, to transform the PDE to the basic Heat Equation,

$$u(x,t) = e^{-[x+(c/2)t]c/2k} v(x,t)$$

so that the PDE for u is transformed to

$$v_t = kv_{xx} \tag{1}$$

The initial condition is

$$v(x,0) = u(x,0)\, e^{xc/2k} = f(x)\, e^{xc/2k} \tag{2}$$

and the BC is

$$0 = \frac{\partial u}{\partial x}(0,t) = e^{-\left(c^2/4k\right)t} \left(-\frac{c}{2k} v(0,t) + \frac{\partial v}{\partial x}(0,t) \right)$$

Thus

$$0 = -\frac{c}{2k} v(0,t) + \frac{\partial v}{\partial x}(0,t) \tag{3}$$

We extend $v(x,t)$ to the infinite rod $-\infty < x < \infty$, and let's suppose the IC is $v(x,0) = \tilde{f}(x)$. The solution to the PDE (1) and the IC is, from class,

$$v(x,t) = \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-s)^2}{4kt}\right) ds$$

We now have to choose $\tilde{f}(x)$ to satisfy the BC (3). First, compute the following:

$$v(0,t) = \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{\sqrt{4\pi kt}} \exp\left(-\frac{s^2}{4kt}\right) ds$$

$$v_x(0,t) = \int_{-\infty}^{\infty} \frac{s\tilde{f}(s)}{2kt\sqrt{4\pi kt}} \exp\left(-\frac{s^2}{4kt}\right) ds$$

Thus

$$-\frac{c}{2k} v(0,t) + \frac{\partial v}{\partial x}(0,t) = \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{2k\sqrt{4\pi kt}} \left(-c + \frac{s}{t}\right) \exp\left(-\frac{s^2}{4kt}\right) ds \tag{4}$$

So if we define

$$\tilde{f}(x) = \begin{cases} f(x)\, e^{xc/2k}, & x \geq 0, \\ -f(-x)\, e^{-xc/2k} \frac{-c-x/t}{-c+x/t}, & x < 0, \end{cases}$$

the integrand in (4) is odd, so that

$$-\frac{c}{2k} v(0,t) + \frac{\partial v}{\partial x}(0,t) = 0.$$

Note that $\tilde{f}(x)$ is neither even nor odd, but by choosing it we satisfy the BC (4). Also, for $x > 0$, $\tilde{f}(x) = f(x)\, e^{xc/2k}$, which is the IC (2) for $v(x,t)$. Now with $f(x) = \delta(x-1)$, we have

$$\tilde{f}(x) = \begin{cases} \delta(x-1)\, e^{xc/2k}, & x \geq 0, \\ -\delta(-x-1)\, e^{-xc/2k} \frac{-c-x/t}{-c+x/t}, & x < 0, \end{cases}$$

and hence

$$\begin{aligned}
v(x,t) &= \int_{-\infty}^{\infty} \frac{\tilde{f}(s)}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-s)^2}{4kt}\right) ds \\
&= -\int_{-\infty}^{0} \frac{\delta(-s-1)\, e^{-sc/2k}}{\sqrt{4\pi kt}} \frac{-c-s/t}{-c+s/t} \exp\left(-\frac{(x-s)^2}{4kt}\right) ds \\
&\quad + \int_{0}^{\infty} \frac{\delta(s-1)\, e^{sc/2k}}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-s)^2}{4kt}\right) ds \\
&= -\int_{-\infty}^{0} \frac{\delta(-s-1)\, e^{-sc/2k}}{\sqrt{4\pi kt}} \frac{-c-s/t}{-c+s/t} \exp\left(-\frac{(x-s)^2}{4kt}\right) ds \\
&\quad + \int_{0}^{\infty} \frac{\delta(s-1)\, e^{sc/2k}}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-s)^2}{4kt}\right) ds \\
&= \frac{e^{c/2k}}{\sqrt{4\pi kt}} \left(-\frac{c-1/t}{c+1/t} \exp\left(-\frac{(x+1)^2}{4kt}\right) + \exp\left(-\frac{(x-1)^2}{4kt}\right)\right)
\end{aligned}$$

Thus

$$\begin{aligned}
u(x,t) &= e^{-[x+(c/2)t]c/2k} v(x,t) \\
&= \frac{e^{-[x+(c/2)t+1]c/2k}}{\sqrt{4\pi kt}} \left(-\frac{c-1/t}{c+1/t} \exp\left(-\frac{(x+1)^2}{4kt}\right) + \exp\left(-\frac{(x-1)^2}{4kt}\right)\right)
\end{aligned}$$

is the solution of the Heat Equation with Convection on the semi-infinite rod, insulated at $x = 0$. Plots are given in Figure 2.

2 Problem 2

Do problem 10.6.4 in Haberman (p 499-500), both (a) and (b). The answer for (a) is in the back - please show how to get that answer. You may find sections 10.5 and 10.6 in Haberman useful as reference reading.

Solutions: Solve Laplace's equation on the half plane,

$$\nabla^2 u = 0, \qquad x > 0, \qquad y > 0$$

subject to the BCs

$$u(0,y) = 0$$

and either (a)

$$\frac{\partial u}{\partial y}(x,0) = f(x)$$

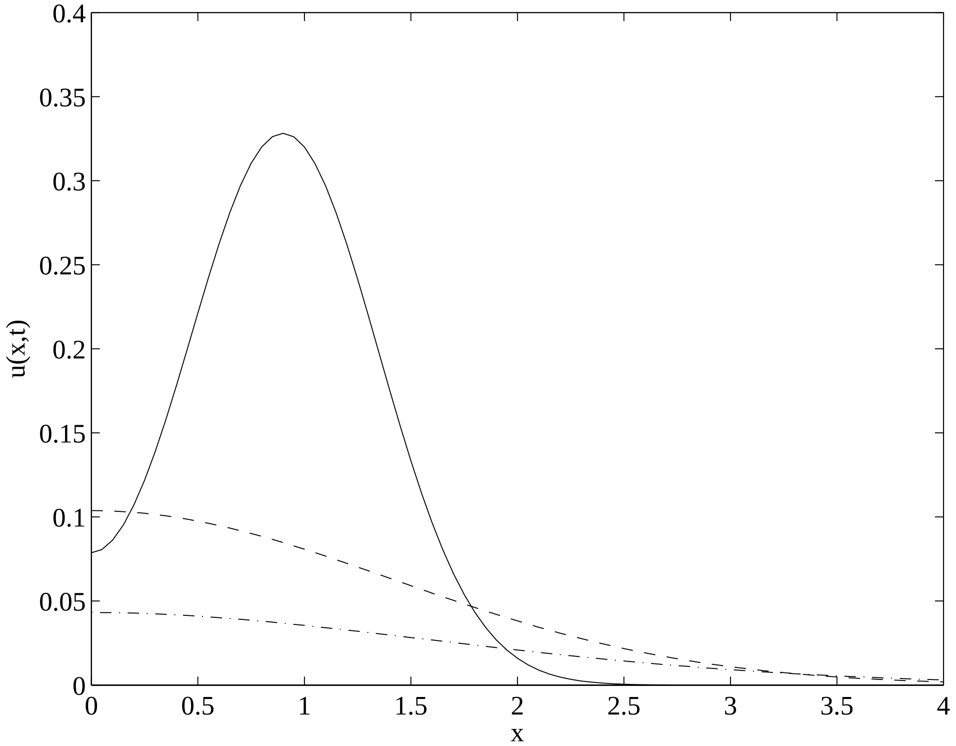

Figure 2: Sketch of $u(x,t)$ with $c = k$, for $kt = 0.1$ (solid), 1 (dashed) and 2 (dash-dot).

or (b)

$$u(x,0) = f(x)$$

Since $u = 0$ along $y = 0$, we must extend $f(x)$ to be odd,

$$\tilde{f}(x) = \begin{cases} f(x), & x \geq 0, \\ -f(-x), & x < 0. \end{cases}$$

We now solve Laplace's equation on the half plane $\{y \geq 0, -\infty < x < \infty\}$, as in §3 of the Notes,

$$\begin{aligned} \nabla^2 \tilde{u} &= 0, \qquad -\infty < x < \infty, \qquad y > 0 \\ \tilde{u}(x,0) &= \tilde{f}(x), \qquad -\infty < x < \infty, \\ \tilde{u}(0,y) &= 0, \qquad y > 0 \end{aligned}$$

Since the inhomogeneous BC is imposed along the x-axis, we employ the Fourier Transform in x,

$$\mathcal{F}[g(x,y)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x,y) e^{i\omega x} dx$$

and define $\bar{U}(\omega, y) = \mathcal{F}[\tilde{u}(x, y)]$. As before, we have

$$\mathcal{F}[\tilde{u}_{xx}] = -\omega^2 \mathcal{F}[\tilde{u}] = -\omega^2 \bar{U}(\omega, y), \qquad \mathcal{F}[\tilde{u}_{yy}] = \frac{\partial^2}{\partial y^2}\mathcal{F}[\tilde{u}] = \frac{\partial^2}{\partial y^2}\bar{U}(\omega, y).$$

Hence Laplace's equation becomes

$$\frac{\partial^2}{\partial y^2}\bar{U}(\omega, y) - \omega^2 \bar{U}(\omega, y) = 0$$

Solving the ODE and being careful about the fact that ω can be positive or negative, we have

$$\bar{U}(\omega, y) = c_1(\omega) e^{-|\omega| y} + c_2(\omega) e^{|\omega| y}$$

where $c_1(\omega)$, $c_2(\omega)$ are arbitrary functions. Since the temperature must remain bounded as $y \to \infty$, we must have $c_2(\omega) = 0$. Thus

$$\bar{U}(\omega, y) = c_1(\omega) e^{-|\omega| y} \tag{5}$$

(a) Imposing the BC at $y = 0$ gives

$$-|\omega| c_1(\omega) = \frac{\partial}{\partial y}\bar{U}(\omega, y)\bigg|_{y=0} = \mathcal{F}\left[\frac{\partial}{\partial y}\tilde{u}(x, 0)\right] = \mathcal{F}[f(x)]$$

Thus

$$\bar{U}(\omega, y) = \mathcal{F}[f(x)] \frac{e^{-|\omega| y}}{-|\omega|}$$

Note that the IFT of $e^{-|\omega| y} / (-|\omega|)$ is

$$\begin{aligned}
\mathcal{F}^{-1}\left[\frac{e^{-|\omega| y}}{-|\omega|}\right] &= \int_{-\infty}^{\infty} \frac{e^{-|\omega| y}}{-|\omega|} e^{-i\omega x} d\omega \\
&= \int_{-\infty}^{\infty} \left(\int e^{-|\omega| y} dy\right) e^{-i\omega x} d\omega \\
&= \int \left(\int_{-\infty}^{\infty} e^{-|\omega| y} e^{-i\omega x} d\omega\right) dy \\
&= \int \mathcal{F}^{-1}\left[e^{-|\omega| y}\right] dy
\end{aligned}$$

In the text and in section 3 of the notes, we showed that

$$\mathcal{F}^{-1}\left[e^{-|\omega| y}\right] = \frac{2y}{x^2 + y^2}$$

Thus

$$\mathcal{F}^{-1}\left[\frac{e^{-|\omega| y}}{-|\omega|}\right] = \int \left(\frac{2y}{x^2 + y^2}\right) dy = \ln\left(x^2 + y^2\right)$$

Therefore, applying the Convolution Theorem with $\mathcal{F}^{-1}\left[c_1(\omega)\right] = \tilde{f}(x)$ and $\mathcal{F}^{-1}\left[e^{-|\omega|y}/\left(-|\omega|\right)\right]$ gives

$$\begin{aligned}
\tilde{u}(x,y) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(s)\ln\left((x-s)^2+y^2\right)ds \\
&= \frac{1}{2\pi}\int_{-\infty}^{0}\tilde{f}(s)\ln\left((x-s)^2+y^2\right)ds + \frac{1}{2\pi}\int_{0}^{\infty}\tilde{f}(s)\ln\left((x-s)^2+y^2\right)ds \\
&= -\frac{1}{2\pi}\int_{-\infty}^{0}f(-s)\ln\left((x-s)^2+y^2\right)ds + \frac{1}{2\pi}\int_{0}^{\infty}f(s)\ln\left((x-s)^2+y^2\right)ds \\
&= \frac{1}{2\pi}\int_{\infty}^{0}f(s)\ln\left((x+s)^2+y^2\right)ds + \frac{1}{2\pi}\int_{0}^{\infty}f(s)\ln\left((x-s)^2+y^2\right)ds \\
&= \frac{1}{2\pi}\int_{0}^{\infty}f(s)\ln\frac{(x-s)^2+y^2}{(x+s)^2+y^2}ds
\end{aligned}$$

(b) Imposing the BC at $y=0$ gives

$$c_1(\omega) = \bar{U}(\omega,0) = \mathcal{F}\left[\tilde{u}(x,0)\right] = \mathcal{F}\left[\tilde{f}(x)\right].$$

Therefore, applying the Convolution Theorem with $\mathcal{F}^{-1}\left[c_1(\omega)\right] = \tilde{f}(x)$ and $\mathcal{F}^{-1}\left[e^{-|\omega|y}\right] = 2y/\left(x^2+y^2\right)$ gives

$$\tilde{u}(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{f}(s)\frac{2y}{(x-s)^2+y^2}ds$$

In both (a) and (b), limiting $x \geq 0$ gives the solution to Laplace's equation on the quarter plane,

$$u(x,y) = \tilde{u}(x,y), \qquad x \geq 0.$$

You don't have to, but you can rearrange this some more,

$$\begin{aligned}
u(x,y) &= \frac{-1}{2\pi}\int_{-\infty}^{0}f(-s)\frac{2y}{(x-s)^2+y^2}ds + \frac{1}{2\pi}\int_{0}^{\infty}f(s)\frac{2y}{(x-s)^2+y^2}ds \\
&= \frac{1}{2\pi}\int_{\infty}^{0}f(s)\frac{2y}{(x+s)^2+y^2}ds + \frac{1}{2\pi}\int_{0}^{\infty}f(s)\frac{2y}{(x-s)^2+y^2}ds \\
&= \frac{y}{\pi}\int_{0}^{\infty}f(s)\left(\frac{-1}{(x+s)^2+y^2}+\frac{1}{(x-s)^2+y^2}\right)ds \\
&= \frac{4xy}{\pi}\int_{0}^{\infty}\frac{sf(s)\,ds}{\left((x+s)^2+y^2\right)\left((x-s)^2+y^2\right)}
\end{aligned}$$