

# Problems for Quasi-Linear PDEs

## 18.303 Linear Partial Differential Equations

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Fall 2006

### 1 Problem 1 : Traffic flow

Solve the traffic flow problem

$$\frac{\partial u}{\partial t} + (1 - 2u) \frac{\partial u}{\partial x} = 0, \quad u(x, 0) = f(x)$$

for an initial traffic group

$$f(x) = \begin{cases} \frac{1}{4}, & |x| \geq 2 \\ 1 - \frac{3}{8}|x|, & |x| < 2 \end{cases}$$

(a) At what time  $t_s$  and position  $x_s$  does a shock first form?

(b) Identify the important values of  $s$  and find the corresponding characteristics.

Sketch the characteristics and indicate the region in the  $xt$ -plane in which the solution is well-defined (i.e. does not break down).

(c) Construct tables as in class for the  $x$  and  $u$  values at the important values of  $s$  for times  $t = 0, 2/3, 4/3$ . Use these tables to sketch the density profile  $u = u(x, t)$  vs.  $x$  for  $t = 0, 2/3, 4/3$ .

### 2 Problem 2 : Water waves

The surface displacement for shallow water waves is governed by (in scaled coordinates),

$$\left(1 + \frac{3}{2}h\right) \frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} = 0$$

Here,  $h = 0$  is the mean free surface of the water. Consider the initial water wave profile

$$h(x, 0) = f(x) = \begin{cases} a \sin x, & 0 \leq x \leq \pi \\ 0, & x < 0, \quad x > \pi \end{cases} \quad (1)$$

where  $a > 0$  is constant.

(a) Find the parametric solution and characteristic curves.

(b) Show that two characteristics starting at  $s = s_1$  and  $s = s_2$  where  $s_1, s_2 \in (\pi/2, \pi)$  intersect at time

$$t_{int} = -\frac{2}{3a} \frac{s_1 - s_2}{\sin s_1 - \sin s_2}$$

Show that

$$t_{int} \geq \frac{2}{3a}, \quad \text{for all } s_1, s_2 \in (\pi/2, \pi)$$

and

$$t_{int} \rightarrow \frac{2}{3a}, \quad \text{as } s_1, s_2 \rightarrow \pi$$

Thus the solution breaks down along the characteristics starting at  $s = \pi$ , when  $t = t_c = 2/(3a)$ . At what position  $x$  does the solution break down?

(c) Calculate  $\partial h / \partial x$  using implicitly differentiation (the solution cannot be found explicitly) and hence show that along the characteristic starting at  $s = \pi$ ,

$$\lim_{t \rightarrow t_c^-} \frac{\partial h}{\partial x} = -\infty$$

Thus the wave slope becomes vertical.

(d) Plot the wave profile  $h(x, t_c)$  with  $a = 1$ . Since the initial waveform is not piecewise linear, we must choose more than a few points in order to plot the solution. Choose the following 11 values of  $s$  equally spaced from 0 to  $\pi$ :

$$s_n = \frac{n}{10}\pi, \quad n = 0, 1, 2, \dots, 10 \quad (2)$$

and construct a table at  $t = t_c$ , as we did in the notes, for  $s$ ,  $h$  and  $x$ . Then plot  $h$  vs.  $x$  and draw a smooth curve through the points to obtain the wave profile  $h(x, t_c)$ . Obviously if you use Matlab you can use 100 points or more to make the plot very smooth. Be sure to label where the wave is vertical and where the maximum displacement occurs. Plot the initial profile  $h(x, 0)$  on the same plot for comparison.

### 3 Problem 3

Consider the quasi-linear PDE and initial condition

$$\begin{aligned} u_t + 2u u_x + 3u &= 0, & t > 0, & \quad -\infty < x < \infty \\ u(x, 0) &= b \sin x, & & \quad -\infty < x < \infty \end{aligned}$$

where  $b > 0$  is constant.

- (a) Find the parametric solution and characteristic curves.  
 (b) Show that the solution  $u$  can be written in the following implicit form

$$\sin\left(x - \frac{2}{3}u(e^{3t} - 1)\right) = \frac{1}{b}ue^{3t}. \quad (3)$$

(c) For  $b = 2$ , show that the solution first breaks down at  $t = t_c = (1/3)\ln 4 = (2/3)\ln 2$ . Along the characteristic through  $(x, t) = (\pi, 0)$ , find an expression for  $u_x$  and show that the left limit of  $u_x$  as  $t \rightarrow t_c^-$  is

$$\lim_{t \rightarrow t_c^-} u_x = -\infty.$$

Hint: differentiate (3) implicitly with respect to  $x$  and set  $(x, t) = (\pi, 0)$ .

(d) For  $b = 2$ , sketch the characteristics and the solution profile  $u(x, t_c)$  at time  $t_c$ . Note: since the initial condition is  $2\pi$ -periodic, so too will be the solution, so you only have to plot one period of  $s$  values. Use the interval  $s \in [0, 2\pi]$ . When sketching the solution choose the following values for  $s$ ,

$$s_n = \frac{n}{5}\pi, \quad n = 0, 1, 2, \dots, 10$$

and construct a table at  $t = t_c$ , as we did in the notes, for  $s$ ,  $u$  and  $x$ . Then plot  $u$  vs.  $x$  and draw a smooth curve through the points to obtain the wave profile  $u(x, t_c)$ . Obviously if you use Matlab you can use 100 points or more to make the plot very smooth. Be sure to label where the wave is vertical and where the maximum displacement occurs. Plot the initial profile  $u(x, 0)$  on the same plot for comparison.

(e) Show that the solution exists for all time if  $0 < b \leq 3/2$ .