Solutions to Problems for Quasi-Linear PDEs

18.303 Linear Partial Differential Equations

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1 Problem 1

Solve the traffic flow problem

$$\frac{\partial u}{\partial t} + (1-2u)\frac{\partial u}{\partial x} = 0, \qquad u(x,0) = f(x)$$

for an initial traffic group

$$f(x) = \begin{cases} \frac{1}{4}, & |x| \geq 2 \\ 1 - \frac{3}{8}|x|, & |x| < 2 \end{cases}$$

(a) At what time t_s and position x_s does a shock first form?

(b) Identify the important values of s and find the corresponding characteristics. Sketch the characteristics and indicate the region in the xt-plane in which the solution is well-defined (i.e. does not break down).

(c) Construct tables as in class for the x and u values at the important values of s for times $t = 0,\ 2/3,\ 4/3$. Use these tables to sketch the density profile $u = u(x,t)$ vs. x for $t = 0,\ 2/3,\ 4/3$.

Solution: (a) We can rewrite the PDE as

$$(1-2u, 1, 0) \cdot \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, -1\right) = 0$$

We write t, x and u as functions of $(r;s)$, i.e. $t(r;s)$, $x(r;s)$, $u(r;s)$. We have written $(r;s)$ to indicate r is the variable that parametrizes the curve, while s is a parameter that indicates the position of the particular trajectory on the initial curve. Thus, the parametric solution is

$$\frac{dt}{dr} = 1, \qquad \frac{dx}{dr} = 1 - 2u, \qquad \frac{du}{dr} = 0$$

with initial condition on $r = 0$,

$$t(0; s) = 0, \qquad x(0; s) = s, \qquad u(0; s) = f(s).$$

where $s \in \mathbb{R}$. We find t and u first, since these can be found independently from one another. Integrating the ODEs and imposing the IC for t and u gives

$$t(r; s) = r, \qquad u(r; s) = f(s). \tag{1}$$

Substituting for $u(r; s)$ into the ODE for $x(r; s)$ and integrating gives

$$x(r; s) = (1 - 2f(s))\, r + const$$

Imposing the IC $x(0; s) = s$ gives

$$x(r; s) = (1 - 2f(s))\, r + s. \tag{2}$$

Combining (1) and (2), the characteristics are

$$x = (1 - 2f(s))\, t + s = \begin{cases} \frac{1}{2}t + s, & |s| > 2 \\ \left(\frac{3}{4}|s| - 1\right) t + s, & |s| \leq 2 \end{cases} \tag{3}$$

The first shock occurs at time

$$t_{sh} = \frac{1}{2 \max\{f'(s)\}} = \frac{1}{2\left(\frac{3}{8}\right)} = \frac{4}{3} \tag{4}$$

where the characteristics starting from $s = -2$ and $s = 0$ meet. We can find the shock location using any of $-2 \leq s \leq 0$,

$$x_{sh} = (1 - 2f(0))\, t_{sh} + 0 = -t_s = -\frac{4}{3}.$$

(b) The important values of s are the points of intersection of the lines composing the function $f(s) : s = -2, 0, 2$. The corresponding characteristics are, from (3),

$$\begin{aligned} s &= -2 : x = \frac{t}{2} - 2 \\ s &= 0 : x = -t \\ s &= 2 : x = \frac{t}{2} + 2 \end{aligned} \tag{5}$$

These characteristics are plotted as solid lines in Figure 1. Other characteristics are plotted as well. The characteristics corresponding to $s = -2, 0, 2$ divide the xt-plane into four regions. The solution is valid over the time interval $0 \leq t < 4/3$. At $t = t_{sh} = 4/3$ the characteristics with $-2 \leq s \leq 0$ meet.

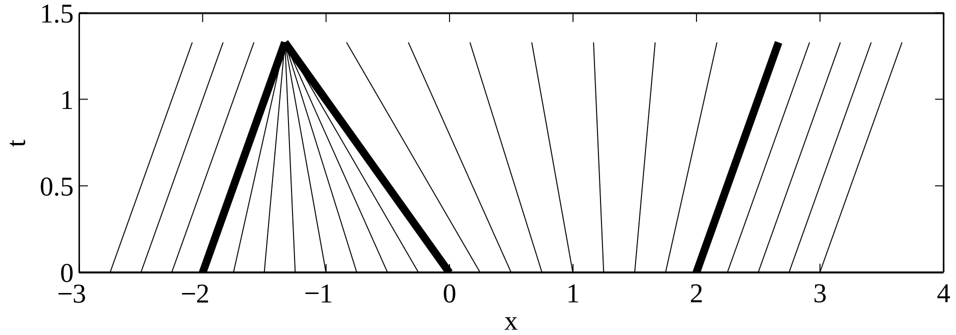

Figure 1: Sketch of characteristics up to the shock time $t = 4/3$. Thick lines are important characteristics.

(c) We construct tables, as in class, nothing that $u = f(s)$ and x is given in (5):

$$t = 2/3: \quad \begin{array}{llll} s & -2 & 0 & 2 \\ u & 1/4 & 1 & 1/4 \\ x & -5/3 & -2/3 & 7/3 \end{array}$$

$$t = 4/3: \quad \begin{array}{llll} s & -2 & 0 & 2 \\ u & 1/4 & 1 & 1/4 \\ x & -4/3 & -4/3 & 8/3 \end{array}$$

Notice that since $u = f(s)$ your u values are the same in each table. For more complicated systems, u might depend on t, so the table will still work for those cases. For each time, the points are plotted in Figure 2 and the dots are connected. This gives the density profile $u(x,t)$ for $t = 2/3$ and $t = 4/3$.

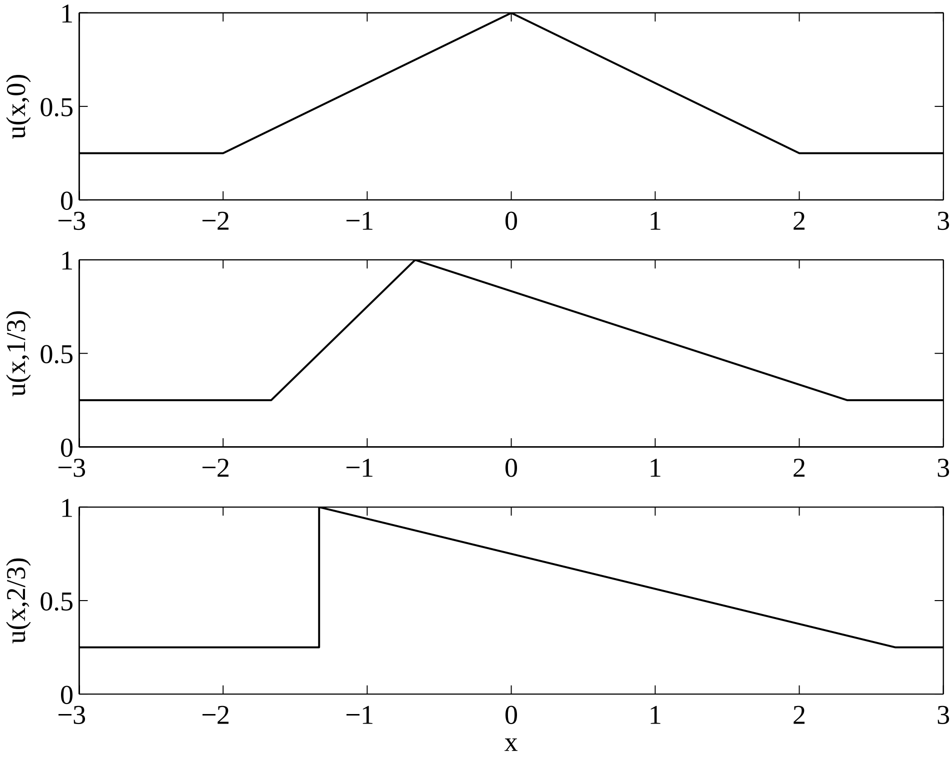

Figure 2: Sketch of density profiles $u = u(x,t)$ vs. x at times $t = 0$, $2/3$ and $t = 4/3$.

2 Problem 2 : Water waves

The surface displacement for shallow water waves is governed by (in scaled coordinates),

$$\left(1 + \frac{3}{2}h\right)\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} = 0$$

Here, $h = 0$ is the mean free surface of the water. Consider the initial water wave profile

$$h(x,0) = f(x) = \begin{cases} a\sin x, & 0 \le x \le \pi \\ 0, & x < 0, \quad x > \pi \end{cases} \tag{6}$$

(a) Find the parametric solution and characteristic curves.

Solution: We can rewrite the PDE as

$$\left(1 + \frac{3}{2}h, 1, 0\right) \cdot (h_x, h_t, -1) = 0$$

and hence the parametric solution are given by

$$\frac{\partial x}{\partial r} = 1 + \frac{3}{2}h, \qquad \frac{\partial t}{\partial r} = 1, \qquad \frac{\partial h}{\partial r} = 0,$$

with initial conditions $t(0) = 0$, $x(0) = s$ and $h(x, 0) = h(s, 0) = f(s)$. Solving the ODEs subject to the initial conditions gives the parametric solution

$$t = r, \qquad h = f(s), \qquad x = \left(1 + \frac{3}{2} f(s)\right) r + s \tag{7}$$

for $s \in \mathbb{R}$. Since $t = r$, we can also write

$$x = \left(1 + \frac{3}{2} f(s)\right) t + s$$

(b) Show that two characteristics starting at $s = s_1$ and $s = s_2$ where $s_1, s_2 \in (\pi/2, \pi)$ intersect at time

$$t_{int} = -\frac{2}{3a} \frac{s_1 - s_2}{\sin s_1 - \sin s_2}$$

Show that

$$t_{int} \geq \frac{2}{3a}, \qquad \text{for all } s_1, s_2 \in (\pi/2, \pi)$$

and

$$t_{int} \to \frac{2}{3a}, \qquad \text{as } s_1, s_2 \to \pi$$

Thus the solution breaks down along the characteristics starting at $s = \pi$, when $t = t_c = 2/(3a)$. At what position x does the solution break down?

Solution: From (7), the solutions starting at $s = s_1$ and $s = s_2$ where $s_1, s_2 \in (0, \pi)$ (and, without loss of generality, $s_1 < s_2$) intersect when

$$\left(1 + \frac{3}{2} f(s_1)\right) t_{int} + s_1 = x_{int} = \left(1 + \frac{3}{2} f(s_2)\right) t_{int} + s_2$$

Solving for the time t_{int} gives

$$t_{int} = \frac{2}{3} \frac{s_2 - s_1}{f(s_1) - f(s_2)}$$

Since $s_1, s_2 \in (0, \pi)$, substituting for $f(s)$ from (6) gives

$$\begin{aligned} t_{int} &= \frac{2}{3} \frac{s_2 - s_1}{a \sin s_1 - a \sin s_2} \\ &= -\frac{2}{3a} \left(\frac{s_1 - s_2}{\sin s_1 - \sin s_2}\right) \end{aligned} \tag{8}$$

By the mean value theorem,

$$\sin s_1 - \sin s_2 = (s_1 - s_2) \cos \xi$$

for some $\xi \in [s_1, s_2] \subseteq (\pi/2, \pi)$, so that (8) becomes

$$t_{int} = -\frac{2}{3a} \frac{1}{\cos \xi} \tag{9}$$

For $\xi \in [s_1, s_2] \subseteq (\pi/2, \pi)$, $-1 < \cos\xi < 0$ so that (9) becomes

$$t_{int} = -\frac{2}{3a}\frac{1}{\cos\xi} \geq \frac{2}{3a}$$

Note that as s_1, $s_2 \to \pi$, ξ also approaches π and hence from (9),

$$\lim_{s_1, s_2 \to \pi} t_{int} = \lim_{\xi \to \pi} t_{int} = \frac{2}{3a}$$

This implies that along the characteristic starting at $s = \pi$, the solution breaks down at $t = t_c = 2/(3a)$. The x-value where the breakdown occurs is

$$x = \left(1 + \frac{3}{2} f(\pi)\right)\frac{2}{3a} + \pi = \left(1 + \frac{3a}{2}\sin(\pi)\right)\frac{2}{3a} + \pi = \frac{2}{3a} + \pi.$$

(c) Calculate $\partial h/\partial x$ using implicitly differentiation (the solution cannot be found explicitly) and hence show that along the characteristic starting at $s = \pi$,

$$\lim_{t \to t_c^-} \frac{\partial h}{\partial x} = -\infty$$

Thus the wave slope becomes vertical.

Solution: By the chain rule,

$$\frac{\partial h}{\partial x} = \frac{\partial h}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial h}{\partial s}\frac{\partial s}{\partial x} = 0 + f'(s)\frac{\partial s}{\partial x} = f'(s)\left(\frac{\partial x}{\partial s}\right)^{-1} = \frac{f'(s)}{\frac{3}{2}f'(s)\,t + 1} \tag{10}$$

Note that

$$f'(\pi) = a\cos\pi = -a,$$

and hence

$$\frac{\partial h}{\partial x} = \frac{-a}{-\frac{3}{2}at + 1}$$

Thus, the (left) limit as $t \to t_c^-$ (where $t_c = 2/(3a)$) is

$$\lim_{t \to t_c^-} \frac{\partial h}{\partial x} = \lim_{t \to t_c^-} \frac{-a}{-\frac{3}{2}at + 1} = -\infty$$

(d) Plot the wave profile $h(x, t_c)$ with $a = 1$. Since the initial waveform is not piecewise linear, we must choose more than a few points in order to plot the solution. Choose the following 11 values of s equally spaced from 0 to π:

$$s_n = \frac{n}{10}\pi, \qquad n = 0, 1, 2, ..., 10 \tag{11}$$

and construct a table at $t = t_c$, as we did in the notes, for s, h and x. Then plot h vs. x and draw a smooth curve through the points to obtain the wave profile

$h(x, t_c)$. Obviously if you use Matlab you can use 100 points or more to make the plot very smooth. Be sure to label where the wave is vertical and where the maximum displacement occurs. Plot the initial profile $h(x, 0)$ on the same plot for comparison.

Solution: Note that the extrema of the displacement occurs where $\partial h/\partial x = 0$, or, from (10),

$$\frac{\partial h}{\partial x} = \frac{f'(s)}{\frac{3}{2}f'(s)\,t + 1} = 0 \qquad \Longleftrightarrow \qquad a(-\cos s) = 0 \qquad \Longleftrightarrow \qquad s = \pi/2$$

Thus the maximum displacement occurs along the $s = \pi/2$ characteristic, where $h = f(\pi/2) = a$. For the particular s values in (11), the characteristics are

$$x_n = \left(1 + \frac{3}{2}f(s_n)\right)t + s_n$$

I didn't ask you to plot the characteristics, but these are plotted in figure 3. To find the wave profile at time $t = 2/3$, we construct the table:

$t = \frac{2}{3}$:

s	0	0.1π	0.2π	0.3π	0.4π	0.5π	0.6π	0.7π	0.8π	0.9π	π
h	0	0.31	0.59	0.81	0.95	1	0.95	0.81	0.59	0.31	0
x	0.67	1.29	1.88	2.42	2.87	3.24	3.50	3.67	3.77	3.80	3.81

Plotting a smooth curve through the points h vs. x gives the wave profile $h(x, 2/3) = h(x, t_c)$. Figure 4 illustrates the wave profiles at $t = 0, 1/3, 2/3$, for $a = 1$. We constructed the $t = 1/3$ profile in a similar manner (you were not asked for $t = 1/3$). The profile becomes vertical along the $s = \pi$ characteristic at time $t = 2/3$ at $x = 2/3 + \pi$. The maximum displacement occurs along the $s = \pi/2$ characteristic, and at time $t = 2/3$, this occurs along

$$x = \left(1 + \frac{3}{2}f\left(\frac{\pi}{2}\right)\right)\frac{2}{3} + \frac{\pi}{2} = \frac{5}{3} + \frac{\pi}{2} \approx 3.24.$$

The interpretation of the plot is that after a time $t = 2/3$ (recall $a = 1$), the wave has moved a distance $x = 2/3$, it's tail has gotten longer, and it's front has steepened.

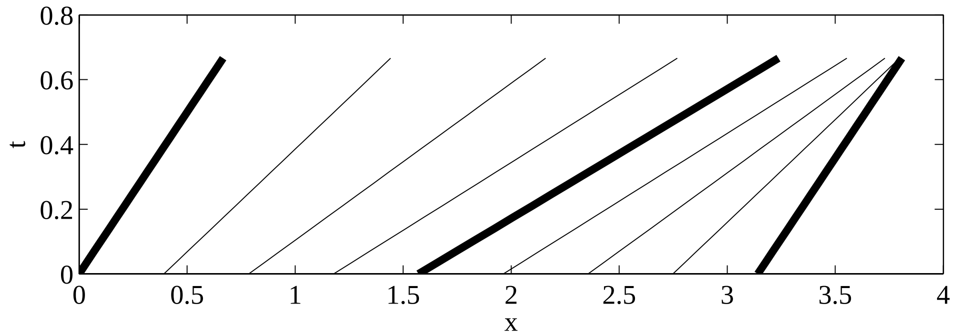

Figure 3: Sketch of characteristics up to the shock time $t = 2/3$. Thick lines are important characteristics. We took $a = 1$.

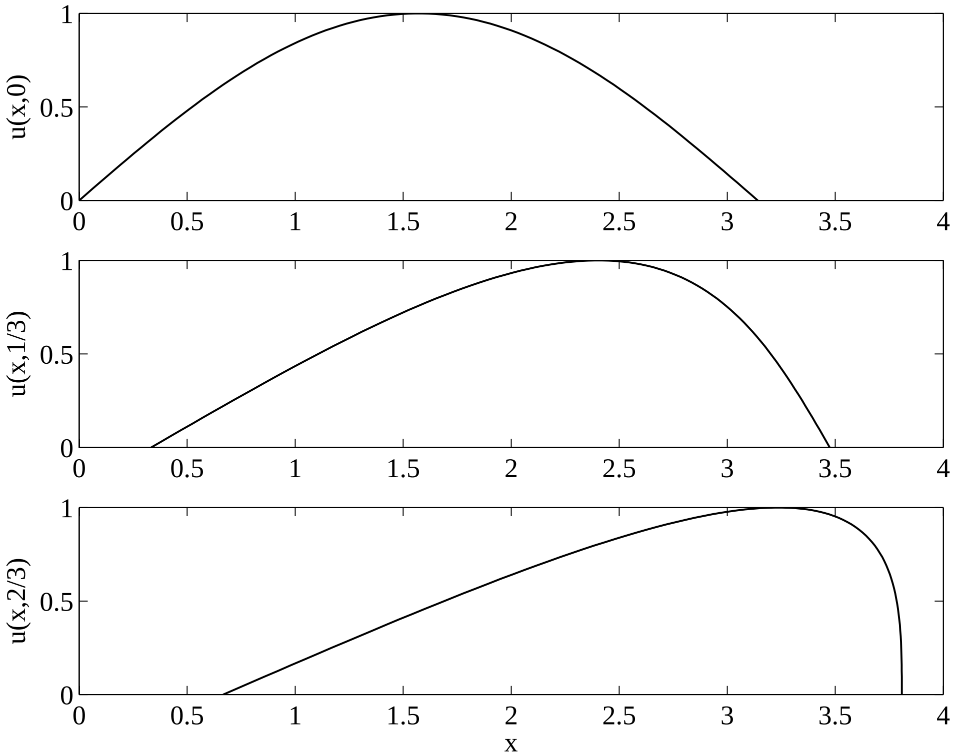

Figure 4: Sketch of wave profiles at times $t = 0,\ 1/3,\ 2/3$. At $t = 2/3$, the wave profile is vertical ($\partial h/\partial x = \infty$ at $x = 2/3 + \pi$, along the $s = \pi$ characteristic. Here, we took $A = 1$.

3 Problem 3

Consider the quasi-linear PDE and initial condition

$$\begin{aligned} u_t + 2u\,u_x + 3u &= 0, \qquad t > 0, \qquad -\infty < x < \infty \\ u(x,0) &= b\sin x, \qquad -\infty < x < \infty \end{aligned}$$

where $A > 0$ is constant.

(a) Find the parametric solution and characteristic curves.

Solution: The PDE can be written as

$$(A, B, C) \cdot (u_x, u_t, -1) = (2u, 1, -3u) \cdot (u_x, u_t, -1) = 0.$$

The characteristic curves are given by

$$\frac{\partial t}{\partial r} = B = 1, \qquad \frac{\partial x}{\partial r} = A = 2u, \qquad \frac{\partial u}{\partial r} = C = -3u$$

The initial conditions at $r = 0$ are $t = 0$, $x = s$, $u = f(s) = b\sin s$. Integrating the ODEs for t and u and imposing the ICs gives

$$t = r, \qquad u = f(s)\,e^{-3r} \tag{12}$$

where $f(s) = b\sin s$. Then the ODE for x becomes

$$\frac{dx}{dr} = 2u = 2f(s)\,e^{-3r}$$

Integrating and imposing the BC $x(r = 0) = s$ gives

$$x = \frac{2}{3} f(s)\left(1 - e^{-3r}\right) + s \tag{13}$$

(b) Show that the solution u can be writtin in the following implicit form

$$\sin\left(x - \frac{2}{3}u\left(e^{3t} - 1\right)\right) = \frac{1}{b} u e^{3t}. \tag{14}$$

Solution: From (12) and (13),

$$u = f(s)\,e^{-3t}, \qquad x = \frac{2}{3} f(s)\left(1 - e^{-3t}\right) + s$$

Solving the first equtaion for $f(s)$ gives

$$f(s) = ue^{3t}$$

But $f(s) = b \sin s$ and hence

$$s = \arcsin\left(\frac{1}{b} u e^{3t}\right)$$

Hence

$$x = \frac{2}{3} u \left(e^{3t} - 1\right) + \arcsin\left(\frac{1}{b} u e^{3t}\right)$$

and rearranging gives

$$\sin\left(x - \frac{2}{3} u \left(e^{3t} - 1\right)\right) = \frac{1}{b} u e^{3t}. \tag{15}$$

This gives the solution u implicitly.

(c) For $b = 2$, show that the solution first breaks down at $t = t_c = (1/3)\ln 4$. Along the characteristic through $(x,t) = (\pi, 0)$, find an expression for u_x and show that

$$\lim_{t \to t_c^-} u_x = -\infty.$$

Hint: differentiate (14) implicitly with respect to x and set $(x,t) = (\pi, 0)$.

Solution: The Jacobian is

$$\frac{\partial(x,t)}{\partial(r,s)} = \det\begin{pmatrix} x_r & x_s \\ t_r & t_s \end{pmatrix} = \det\begin{pmatrix} 2u & \frac{2}{3} f'(s)\left(1 - e^{-3r}\right) + 1 \\ 1 & 0 \end{pmatrix} = -\frac{2}{3} f'(s)\left(1 - e^{-3r}\right) - 1$$

The solution breaks down when the Jacobian is zero, or

$$-\frac{2}{3} f'(s)\left(1 - e^{-3r}\right) - 1 = 0$$

Since $r = t$ and $f'(s) = b \cos s$, we have

$$\frac{2}{3} b \cos s \left(1 - e^{-3t}\right) = -1 \tag{16}$$

Note that the breakdown must occur for $t > 0$, since $t = 0$ will not satisfy the above equation. Also, $(1 - e^{-3t}) > 0$ since $t > 0$. Thus the breakdown occurs when $\cos s < 0$ and $t > 0$. The smallest time for breakdown occurs at the most negative value of $\cos s$, i.e., $\cos s = -1$, when

$$1 - \frac{3}{2b} = e^{-3t}$$

or

$$t_c = -\frac{1}{3} \ln\left(1 - \frac{3}{2b}\right)$$

Since $b = 2$, the first breakdown occurs at $t_c = (1/3)\ln 4 = (2/3)\ln 2$.

Note that $(x,t) = (\pi, 0)$ is along the initial curve, so that $s = x = \pi$. You could also plug $(x,t) = (\pi, 0)$ into (13) to find this. Substituting $s = \pi$ into (12) and (13) gives

$$\begin{aligned} u(t;\pi) &= b(\sin\pi)\, e^{-3t} = 0 \\ x(t;\pi) &= 2b(\sin\pi)\left(1 - e^{-t/2}\right) + \pi = \pi \end{aligned}$$

Thus $x = \pi$ and $u = 0$ for all time t along this characteristic. To find u_x, we differentiate (15) implicitly with respect to x, with $b = 2$,

$$\cos\left(x + \frac{2}{3}u\left(1 - e^{3t}\right)\right)\left(1 + \frac{2}{3}u_x\left(1 - e^{3t}\right)\right) = \frac{1}{2}u_x e^{3t}$$

Substituting $x = \pi$ and $u = 0$ gives

$$-\left(1 + \frac{2}{3}u_x\left(1 - e^{3t}\right)\right) = \frac{1}{2}u_x e^{3t}$$

Solving for u_x gives

$$u_x = \frac{6}{e^{3t} - 4}$$

For $s = \pi$, $\cos s = -1$, so that the solution breaks down along this characteristic at $t = t_c = (2/3)\ln 2$. As $t \to t_c^-$, the limit of u_x is

$$\lim_{t \to t_c^-} u_x = \lim_{t \to t_c^-} \frac{6}{e^{3t} - 4} = -\infty$$

(d) For $b = 2$, sketch the characteristics and the solution profile at time t_c. Note: since the initial condition is 2π-periodic, so too will be the solution, so you only have to plot one period of s values. Use the interval $s \in [0, 2\pi]$. When sketching the solution choose the following values for s,

$$s_n = \frac{n}{5}\pi, \qquad n = 0, 1, 2, ..., 10 \tag{17}$$

and construct a table at $t = t_c$, as we did in the notes, for s, u and x. Then plot u vs. x and draw a smooth curve through the points to obtain the wave profile $u(x, t_c)$. Obviously if you use Matlab you can use 100 points or more to make the plot very smooth. Be sure to label where the wave is vertical and where the maximum displacement occurs. Plot the initial profile $u(x, 0)$ on the same plot for comparison.

Solution: Since the initial condition is periodic, we must only plot the region $0 \leq x \leq 2\pi$, $t \geq 0$. The solution is repeated in the other regions $2(n-1)\pi \leq x \leq 2n\pi$, for all integers n. For the particular s values in (17), the characteristics are

$$x_n = \frac{2}{3}f(s_n)\left(1 - e^{-3t}\right) + s_n$$

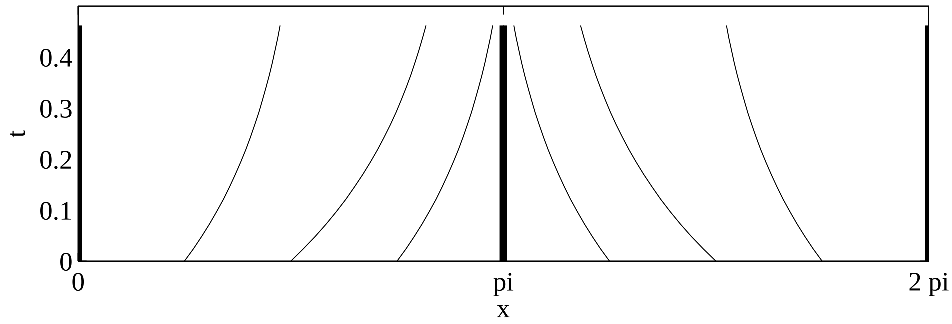

Figure 5: Sketch of characteristics up to the shock time $t = t_c = (2/3)\ln 2$. Thick lines are important characteristics (sides and maximum of wave).

The characteristics are plotted in figure 5. To find the wave profile at time $t = (2/3)\ln 2$, we construct the table:

$t = \frac{2}{3}\ln 2:$	s	0	0.2π	0.4π	0.6π	0.8π	π	1.2π	1.4π	1.6π	1.8π	2π
	u	0	0.29	0.48	0.48	0.29	0	-0.29	-0.48	-0.48	-0.29	0
	x	0	1.22	2.21	2.84	3.10	3.14	3.18	3.45	4.08	5.07	6.28

Characteristics are plotted in Figure 5 up to the time $t = t_c$ and the solution in Figure 6 is plotted at time $t = t_c$.

[Extra comments] Note that $x = \pi$ is a line of symmetry. To see this, consider the characteristics $s = \pi/2$ and $s = 3\pi/2$ with $= 1$,

$$\begin{aligned} x\left(s = \frac{\pi}{2}, t\right) &= \frac{4}{3}\left(1 - e^{-3t}\right) + \frac{\pi}{2} \\ x\left(s = \frac{3\pi}{2}, t\right) &= -\frac{4}{3}\left(1 - e^{-3t}\right) + \frac{3\pi}{2} \\ &= -\left(\frac{4}{3}\left(1 - e^{-3t}\right) + \frac{\pi}{2}\right) + 2\pi \\ &= -x\left(s = \frac{\pi}{2}, t\right) + 2\pi \end{aligned}$$

Thus the characteristics are symmetric about $s = \pi$. Note that $x\left(s = \pi, t\right) = \pi/2$. Thus the characteristics are symmetric about $x = \pi/2$. This is evident in Figure 5.

Note also that the solution can be found implicitly. Substituting $b = 2$ and $t = t_c = (2/3)\ln 2$ into the implicit solution (15) gives

$$\sin\left(x - 2u\right) = 2u$$

and hence

$$x = 2u + \arcsin\left(2u\right)$$

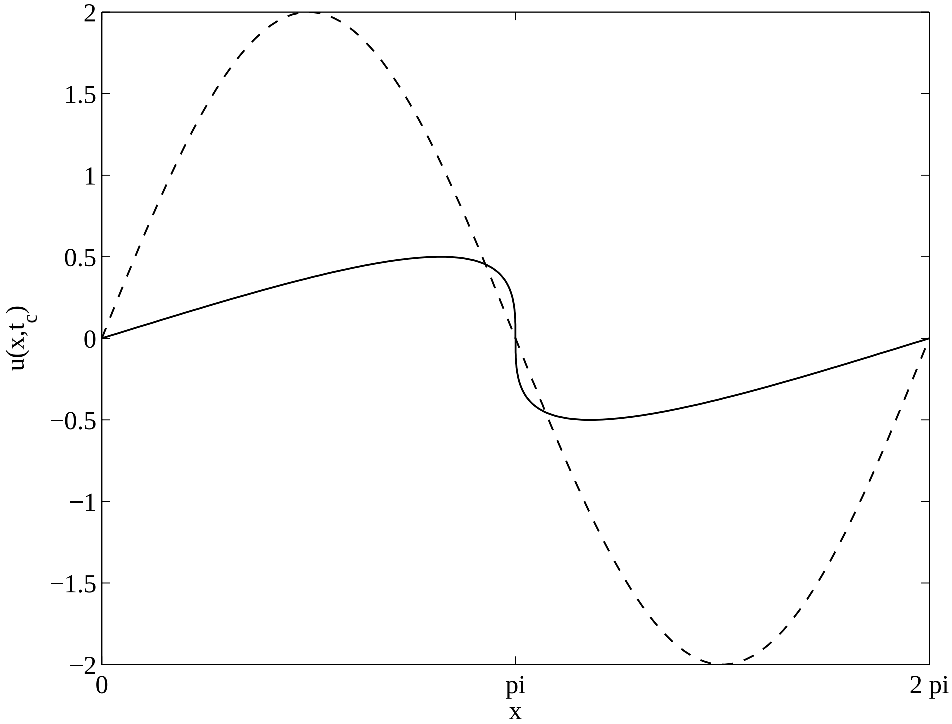

Figure 6: Sketch of $u(x,t_c)$ profile ($t_c = (2/3)\ln 2$, $b = 2$). Since $u(x,t)$ is 2π-periodic in x, the $u(x,t)$ is given by periodicity for values of x outside the region plotted.

Thus by choosing values of u we can plot the corresponding x-values and thus obtain a plot of u vs. x.

(e) Show that the solution exists for all time if $0 < b \leq 3/2$.

Solution: Recall that the solution breaks down if there is an s and t that satisfy Eq. (16),

$$\frac{2}{3}b\left(\cos s\right)\left(1 - e^{-3t}\right) = -1$$

For $0 < b \leq 3/2$, we have $0 < (2/3)b \leq 1$ and for $t \geq 0$, $0 \leq 1 - e^{-3t} < 1$, so that

$$\left|\frac{2}{3}b\left(\cos s\right)\left(1 - e^{-3t}\right)\right| < 1$$

Thus Eq. (16) cannot be satisfied, and the solution is valid for all time $t \geq 0$.