Problems for the 1-D Wave Equation

18.303 Linear Partial Differential Equations

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1 Problem 1

(i) Suppose that an "infinite string" has an initial displacement

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-2x, & 0 \le x \le 1/2 \\ 0, & x < -1 \text{ and } x > 1/2 \end{cases}$$

and zero initial velocity $u_t(x,0) = 0$. Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs $u(x,0) = f(x)$ and $u_t(x,0) = 0$ using D'Alembert's formula. Illustrate the nature of the solution by sketching the ux-profiles $y = u(x,t)$ of the string displacement for $t = 0, 1/2, 1, 3/2$.

(ii) Repeat the procedure in (i) for a string that has zero initial displacement but is given an initial velocity

$$u_t(x,0) = g(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \\ 0, & x < -1 \text{ and } x > 1 \end{cases}$$

2 Problem 2

(i) For an infinite string (i.e. we don't worry about boundary conditions), what initial conditions would give rise to a purely forward wave? Express your answer in terms of the initial displacement $u(x,0) = f(x)$ and initial velocity $u_t(x,0) = g(x)$ and their derivatives $f'(x)$, $g'(x)$. Interpret the result intuitively.

(ii) Again for an infinite string, suppose that $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are zero for $|x| > a$, for some real number $a > 0$. Prove that if $t + x > a$ and $t - x > a$, then the displacement $u(x,t)$ of the string is constant. Relate this constant to $g(x)$.

3 Problem 3

Consider a semi-infinite vibrating string. The vertical displacement $u(x,t)$ satisfies

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad x \geq 0, \qquad t \geq 0 \\ u(0,t) &= 0, \qquad t \geq 0 \\ u(x,0) &= f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad x \geq 0, \end{aligned} \tag{1}$$

The BC at infinity is that $u(x,t)$ must remain bounded as $x \to \infty$.

(a) Show that D'Alembert's formula solves (1) when $f(x)$ and $g(x)$ are extended to be odd functions.

(b) Let

$$f(x) = \begin{cases} \sin^2(\pi x), & 1 \leq x \leq 2 \\ 0, & 0 \leq x \leq 1, \qquad x \geq 2 \end{cases}$$

and $g(x) = 0$ for $x \geq 0$. Sketch u vs. x for $t = 0, 1, 2, 3$.

4 Problem 4

The acoustic pressure in an organ pipe obeys the 1-D wave equation (in physical variables)

$$p_{tt} = c^2 p_{xx}$$

where c is the speed of sound in air. Each organ pipe is closed at one end and open at the other. At the closed end, the BC is that $p_x(0,t) = 0$, while at the open end, the BC is $p(l,t) = 0$, where l is the length of the pipe.

(a) Use separation of variables to find the normal modes $p_n(x,t)$.

(b) Give the frequencies of the normal modes and sketch the pressure distribution for the first two modes.

(c) Given initial conditions $p(x,0) = f(x)$ and $p_t(x,0) = g(x)$, write down the general initial boundary value problem (PDE, BCs, ICs) for the organ pipe and determine the series solutions.