Solution to Problems for the 1-D Wave Equation

18.303 Linear Partial Differential Equations

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1 Problem 1

(i) Suppose that an "infinite string" has an initial displacement

$$u(x,0) = f(x) = \begin{cases} x+1, & -1 \le x \le 0 \\ 1-2x, & 0 \le x \le 1/2 \\ 0, & x < -1 \text{ and } x > 1/2 \end{cases}$$

and zero initial velocity $u_t(x,0) = 0$. Write down the solution of the wave equation

$$u_{tt} = u_{xx}$$

with ICs $u(x,0) = f(x)$ and $u_t(x,0) = 0$ using D'Alembert's formula. Illustrate the nature of the solution by sketching the ux-profiles $y = u(x,t)$ of the string displacement for $t = 0, 1/2, 1, 3/2$.

Solution: D'Alembert's formula is

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t) + \int_{x-t}^{x+t} g(s)\, ds\right)$$

In this case $g(s) = 0$ so that

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) \tag{1}$$

The problem reduces to adding shifted copies of $f(x)$ and then plotting the associated $u(x,t)$. To determine where the functions overlap or where $u(x,t)$ is zero, we plot the characteristics $x \pm t = -1$ and $x \pm t = 1/2$ in the space time plane (xt) in Figure 1.

For $t = 0$, (1) becomes

$$u(x,0) = \frac{1}{2}\left(f(x) + f(x)\right) = f(x)$$

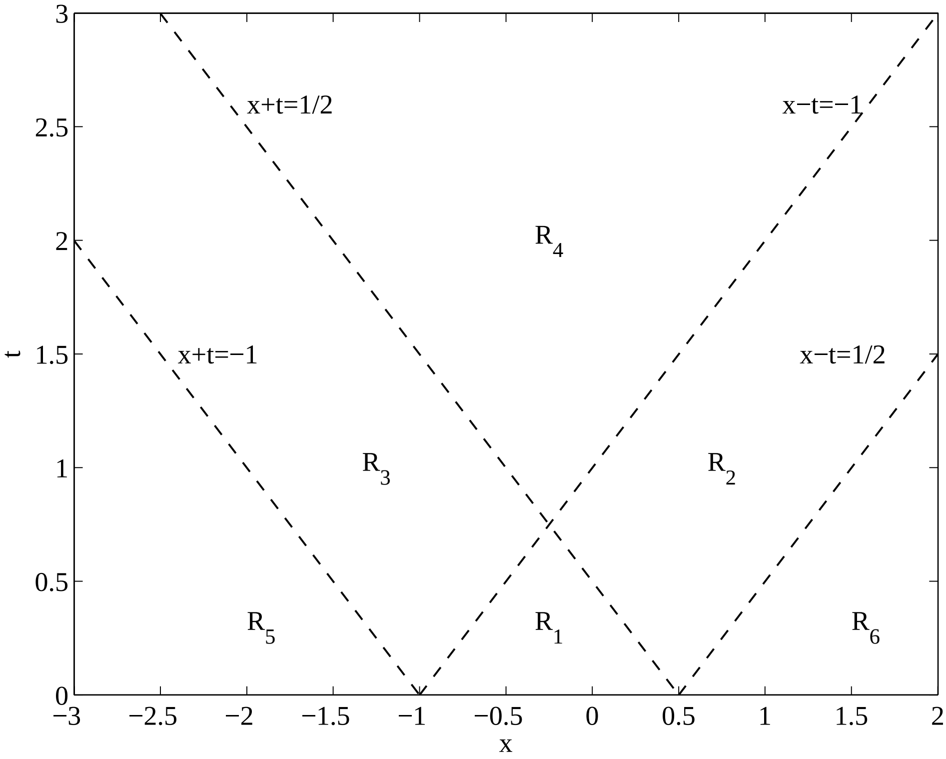

Figure 1: Sketch of characteristics for 1(a).

For $t = 1/2$, (1) becomes

$$u(x,t) = \frac{1}{2}\left(f\left(x - \frac{1}{2}\right) + f\left(x + \frac{1}{2}\right)\right)$$

Note that

$$\begin{aligned} f\left(x - \frac{1}{2}\right) &= \begin{cases} \left(x - \frac{1}{2}\right) + 1, & -1 \le \left(x - \frac{1}{2}\right) \le 0 \\ 1 - 2\left(x - \frac{1}{2}\right), & 0 \le \left(x - \frac{1}{2}\right) \le 1/2 \\ 0, & \left(x - \frac{1}{2}\right) < -1 \text{ and } \left(x - \frac{1}{2}\right) > 1/2 \end{cases} \\ &= \begin{cases} x + \frac{1}{2}, & -\frac{1}{2} \le x \le \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \le x \le 1 \\ 0, & x < -\frac{1}{2} \text{ and } x > 1 \end{cases} \end{aligned}$$

and similarly,

$$f\left(x + \frac{1}{2}\right) = \begin{cases} x + \frac{3}{2}, & -\frac{3}{2} \le x \le -\frac{1}{2} \\ -2x, & -\frac{1}{2} \le x \le 0 \\ 0, & x < -\frac{3}{2} \text{ and } x > 0 \end{cases}$$

Thus, over the region $-\frac{1}{2} \le x \le 0$ we have to be careful about adding the two functions; in the other regions either one or both functions are zero. We have

$$
\begin{aligned}
u\left(x,\frac{1}{2}\right) &= \frac{1}{2}\left(f\left(x-\frac{1}{2}\right)+f\left(x+\frac{1}{2}\right)\right)\\
&= \begin{cases} \frac{x}{2}+\frac{3}{4}, & -\frac{3}{2}\le x\le -\frac{1}{2}\\ -\frac{x}{2}+\frac{1}{4}, & -\frac{1}{2}\le x\le 0\\ \frac{x}{2}+\frac{1}{4}, & 0\le x\le \frac{1}{2}\\ 1-x, & \frac{1}{2}\le x\le 1\\ 0, & x<-\frac{3}{2}\text{ and }x>1 \end{cases}
\end{aligned}
$$

For $t=1$, your plot of the characteristics shows that $f(x-1)$ and $f(x+1)$ do not overlap, so you just have to worry about the different regions. Note that

$$
\begin{aligned}
f(x+1) &= \begin{cases} (x+1)+1, & -1\le x+1\le 0\\ 1-2(x+1), & 0\le x+1\le 1/2\\ 0, & x+1<-1\text{ and }x+1>1/2 \end{cases}\\
&= \begin{cases} x+2, & -2\le x\le -1\\ -1-2x, & -1\le x\le -1/2\\ 0, & x<-2\text{ and }x>-1/2 \end{cases}
\end{aligned}
$$

$$
f(x-1)=\begin{cases} x, & 0\le x\le 1\\ 3-2x, & 1\le x\le 3/2\\ 0, & x<0\text{ and }x>3/2 \end{cases}
$$

so that

$$
\begin{aligned}
u(x,1) &= \frac{1}{2}\left(f(x-1)+f(x+1)\right)\\
&= \begin{cases} \frac{x}{2}+1, & -2\le x\le -1\\ -\frac{1}{2}-x, & -1\le x\le -1/2\\ \frac{x}{2}, & 0\le x\le 1\\ \frac{3}{2}-x, & 1\le x\le 3/2\\ 0, & x<-2,\quad -1/2<x<0,\text{ and }x>3/2 \end{cases}
\end{aligned}
$$

For $t=3/2$, the forward and backward waves are even further apart, and

$$
f\left(x-\frac{3}{2}\right)=\begin{cases} x-\frac{1}{2}, & \frac{1}{2}\le x\le \frac{3}{2}\\ 4-2x, & \frac{3}{2}\le x\le 2\\ 0, & x<\frac{1}{2}\text{ and }x>2 \end{cases}
$$

$$
f\left(x+\frac{3}{2}\right)=\begin{cases} x+\frac{5}{2}, & -\frac{5}{2}\le x\le -\frac{3}{2}\\ -2-2x, & -\frac{3}{2}\le x\le -1\\ 0, & x<-\frac{5}{2}\text{ and }x>-1 \end{cases}
$$

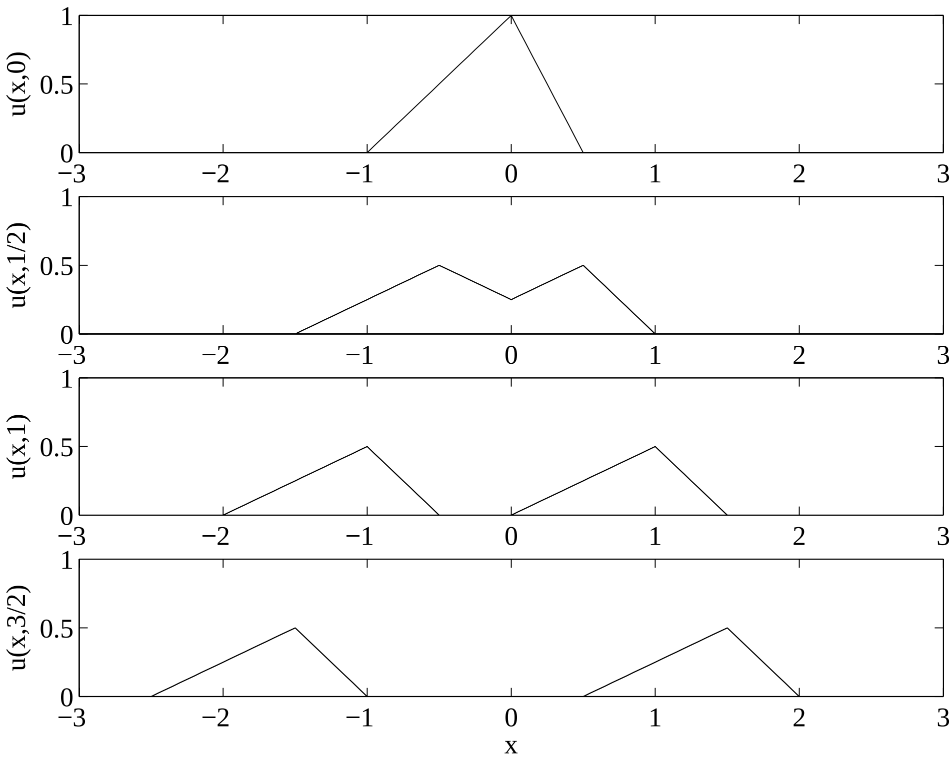

Figure 2: Plot of $u(x, t_0)$ for $t_0 = 0, 1/2, 1, 3/2$ for 1(a).

and hence

$$u\left(x,\frac{3}{2}\right) = \frac{1}{2}\left(f\left(x-\frac{3}{2}\right)+f\left(x+\frac{3}{2}\right)\right)$$

$$= \begin{cases} \frac{x}{2}+\frac{5}{4}, & -\frac{5}{2} \le x \le -\frac{3}{2}, \\ -1-x, & -\frac{3}{2} \le x \le -1, \\ \frac{x}{2}-\frac{1}{4}, & \frac{1}{2} \le x \le \frac{3}{2}, \\ 2-x, & \frac{3}{2} \le x \le 2, \\ 0, & x<-\frac{5}{2},\quad -1<x<\frac{1}{2},\ \text{and } x>2 \end{cases}$$

The solution $u(x, t_0)$ is plotted at times $t_0 = 0, 1/2, 1, 3/2$ in Figure 2. A 3D version of $u(x,t)$ is plotted in Figure 3.

(ii) Repeat the procedure in (i) for a string that has zero initial displacement but is given an initial velocity

$$u_t(x,0) = g(x) = \begin{cases} -1, & -1 \le x < 0 \\ 1, & 0 \le x \le 1 \\ 0, & x<-1 \text{ and } x>1 \end{cases}$$

Solution: D'Alembert's formula is

$$u(x,t) = \frac{1}{2}\left(f(x-t)+f(x+t)+\int_{x-t}^{x+t} g(s)\,ds\right)$$

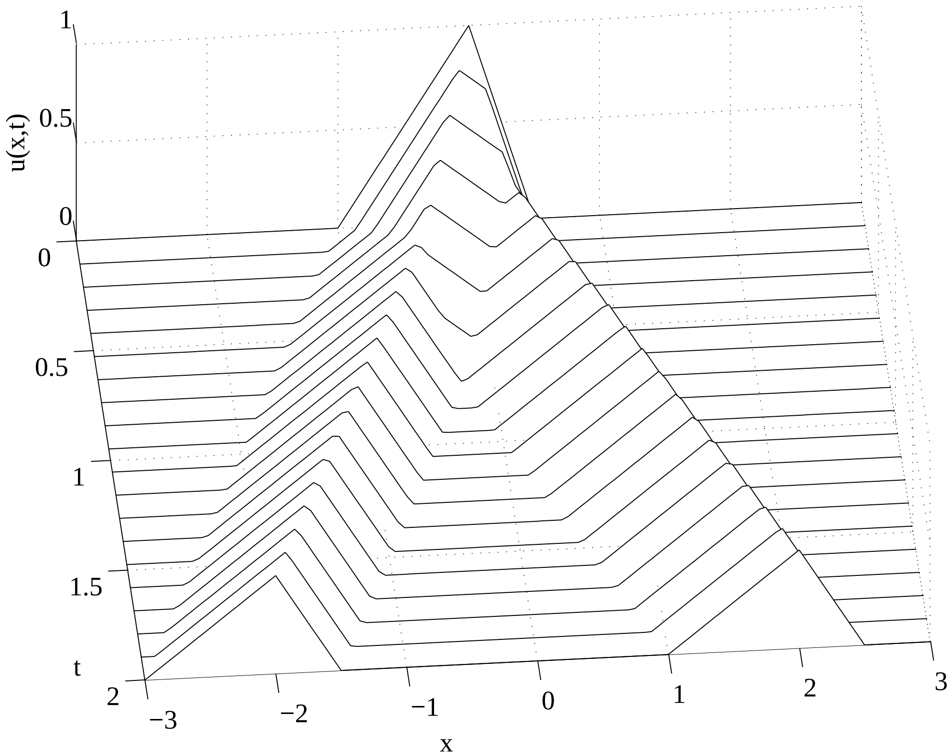

Figure 3: 3D version of $u(x,t)$ for 1(a).

In this case $f(s) = 0$ so that

$$u(x,t) = \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

The problem reduces to noting where $x \pm t$ lie in relation to ± 1 and evaluating the integral. These characteristics are plotted in Figure 1 in the notes.

You can proceed in two ways. First, you can draw two more characterstics $x \pm t = 0$ so you can decide where the integration variable s is with respect to zero, and hence if $g(s) = -1$ or 1. The second way is to note that for $a < b$ and $|a|, |b| < 1$,

$$\int_a^b g(s)\,ds = |b| - |a|$$

for positive and negative a, b. I'll use the second method; the answers you get from the first are the same.

In Region R_1,

$$|x \pm t| \leq 1$$

and hence there are 3 cases: $x-t<0$, x

$$\begin{aligned} u(x,t) &= \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds \\ &= \frac{1}{2}(|x+t|-|x-t|) \end{aligned}$$

In Region R_2, $x+t>1$ and $-1<x-t<1$, so that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\int_{x-t}^{1}+\int_{1}^{x+t}\right) g(s)\,ds = \frac{1}{2}\int_{x-t}^{1} g(s)\,ds \\ &= \frac{1}{2}(1-|x-t|) \end{aligned}$$

In Region R_3, $x-t<-1$ and $-1<x+t<1$, so that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\int_{x-t}^{-1}+\int_{-1}^{x+t}\right) g(s)\,ds = \frac{1}{2}\int_{-1}^{x+t} g(s)\,ds = \frac{1}{2}(|x+t|-|-1|) \\ &= \frac{1}{2}(|x+t|-1) \end{aligned}$$

In Region R_4, $x+t>1$ and $x-t<-1$, so that

$$\begin{aligned} u(x,t) &= \frac{1}{2}\left(\int_{x-t}^{-1}+\int_{-1}^{1}+\int_{1}^{x+t}\right) g(s)\,ds \\ &= \frac{1}{2}\int_{-1}^{1} g(s)\,ds = \frac{1}{2}(-1+1) \\ &= 0 \end{aligned}$$

In Region R_5, $x+t<-1$ and hence $u(x,t)=0$. In region R_6, $x-t>1$, so that $u(x,t)=0$.

At $t=0$,

$$u(x,0) = \frac{1}{2}\int_{x}^{x} g(s)\,ds = 0$$

At $t=1/2$, the regions R_n are given in the notes and

$$u\left(x,\frac{1}{2}\right) = \begin{cases} \frac{1}{2}\left(\left|x+\frac{1}{2}\right|-\left|x-\frac{1}{2}\right|\right), & x\in R_1=\left[-\frac{1}{2},\frac{1}{2}\right] \\ \frac{1}{2}\left(1-\left|x-\frac{1}{2}\right|\right), & x\in R_2=\left[\frac{1}{2},\frac{3}{2}\right] \\ \frac{1}{2}\left(\left|x+\frac{1}{2}\right|-1\right), & x\in R_3=\left[-\frac{3}{2},-\frac{1}{2}\right] \\ 0, & x\in R_5,R_6=\{|x|>3/2\} \end{cases}$$

The absolute values are easy to resolve (i.e. write without them) in this case. For example, for $x\in[-1/2,1/2]$, we have $|x-1/2|=-(x-1/2)$. Thus,

$$u\left(x,\frac{1}{2}\right) = \begin{cases} x, & x\in R_1=\left[-\frac{1}{2},\frac{1}{2}\right] \\ \frac{3}{4}-\frac{x}{2}, & x\in R_2=\left[\frac{1}{2},\frac{3}{2}\right] \\ -\frac{3}{4}-\frac{x}{2}, & x\in R_3=\left[-\frac{3}{2},-\frac{1}{2}\right] \\ 0, & x\in R_5,R_6=\{|x|>3/2\} \end{cases}$$

At $t=1$, the regions R_n are given in the notes and

$$u(x,1)=\begin{cases}\frac{1}{2}(1-|x-1|), & x\in R_2=[0,2],\\ \frac{1}{2}(|x+1|-1), & x\in R_3=[-2,0],\\ 0, & x\in R_5,R_6=\{|x|>3/2\}.\end{cases}$$

You could leave your answer like this, or write it without absolute values (have to divide $[0,2]$ and $[-2,0]$ into cases):

$$u(x,1)=\begin{cases}x/2, & x\in[0,1],\\ \frac{1}{2}(2-x), & x\in[1,2],\\ -\frac{1}{2}(x+2) & x=[-2,-1]\\ x/2, & x=[-1,0],\\ 0, & x\in R_5,R_6=\{|x|>3/2\}.\end{cases}$$

At $t=3/2$, the regions R_n are not given explicitly, but can be found from Figure 1 in the notes by nothing where the line $t=3/2$ crosses each region:

$$u\left(x,\frac{3}{2}\right)=\begin{cases}\frac{1}{2}\left(1-\left|x-\frac{3}{2}\right|\right), & x\in R_2=\left[\frac{1}{2},\frac{5}{2}\right]\\ \frac{1}{2}\left(\left|x+\frac{3}{2}\right|-1\right), & x\in R_3=\left[-\frac{5}{2},-\frac{1}{2}\right]\\ 0, & x\in R_4,R_5,R_6=\{|x|>5/2 \text{ or } |x|<1/2\}\end{cases}$$

Again, you could leave your answer like this, or write it without absolute values (have to divide $[1/2,5/2]$ and $[-5/2,-1/2]$ into cases):

$$u\left(x,\frac{3}{2}\right)=\begin{cases}\frac{1}{2}\left(x-\frac{1}{2}\right), & x\in R_2=\left[\frac{1}{2},\frac{3}{2}\right]\\ \frac{1}{2}\left(\frac{5}{2}-x\right), & x\in R_2=\left[\frac{3}{2},\frac{5}{2}\right]\\ -\frac{1}{2}\left(x+\frac{5}{2}\right), & x\in R_3=\left[-\frac{5}{2},-\frac{3}{2}\right]\\ \frac{1}{2}\left(x+\frac{1}{2}\right), & x\in R_3=\left[-\frac{3}{2},-\frac{1}{2}\right]\\ 0, & x\in R_4,R_5,R_6=\{|x|>5/2 \text{ or } |x|<1/2\}\end{cases}$$

The solution $u(x,t_0)$ is plotted at times $t_0=0,1/2,1,3/2$ in Figure 4.

2 Problem 2

(i) For an infinite string (i.e. we don't worry about boundary conditions), what initial conditions would give rise to a purely forward wave? Express your answer in terms of the initial displacement $u(x,0)=f(x)$ and initial velocity $u_t(x,0)=g(x)$ and their derivatives $f'(x)$, $g'(x)$. Interpret the result intuitively.

Solution: Recall in class that we write D'Alembert's solution as

$$u(x,t)=P(x-t)+Q(x+t) \tag{2}$$

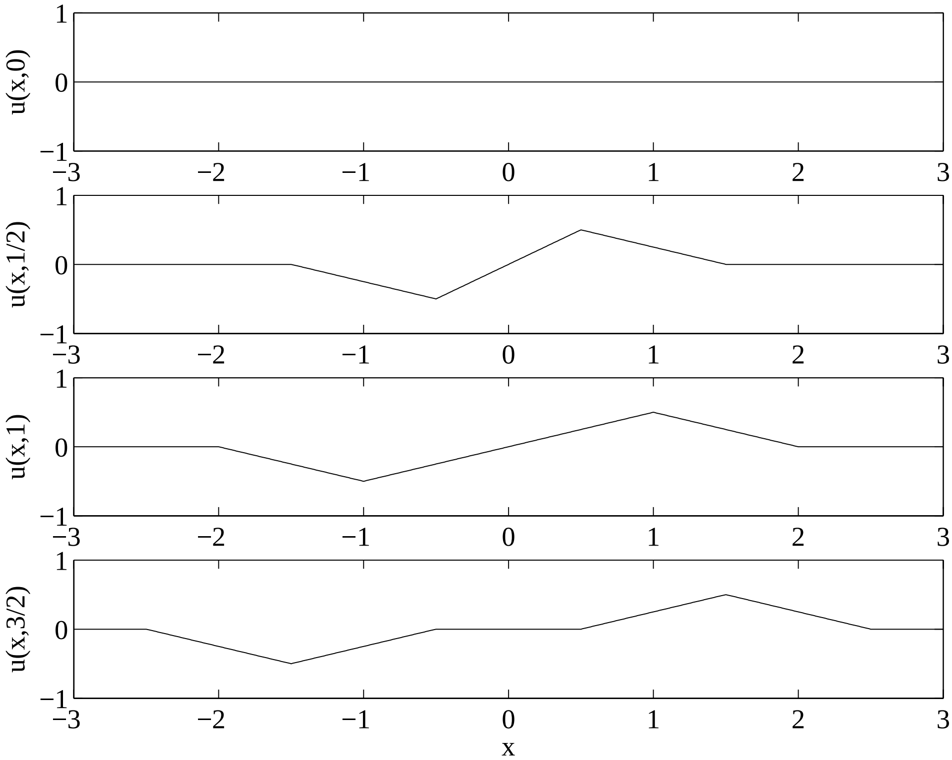

Figure 4: Plot of $u(x, t_0)$ for $t_0 = 0, 1/2, 1, 3/2$ for 1(b).

where

$$Q(x) = \frac{1}{2}\left(f(x) + \int_0^x g(s)\,ds + Q(0) - P(0)\right) \tag{3}$$

$$P(x) = \frac{1}{2}\left(f(x) - \int_0^x g(s)\,ds - Q(0) + P(0)\right) \tag{4}$$

To only have a forward wave, we must have

$$Q(x) = const = q_1$$

Substituting (3) gives

$$Q(x) = q_1 = \frac{1}{2}\left(f(x) + \int_0^x g(s)\,ds + Q(0) - P(0)\right)$$

Differentiating in x gives

$$0 = \frac{1}{2}\left(\frac{df}{dx} + g(x)\right)$$

Thus

$$g(x) = -\frac{df}{dx} \tag{5}$$

Substituting (5) into (3) gives

$$Q(x) = \frac{1}{2}\left(f(0) + Q(0) - P(0)\right)$$

and setting $x = 0$ yields $f(0) - P(0) = Q(0)$. Substituting this and (5) into (4) gives

$$P(x) = \frac{1}{2}\left(2f(x) - f(0) - Q(0) + P(0)\right) = f(x)$$

and hence

$$u(x,t) = f(x-t).$$

The displacement $u(x,t)$ only contains the forward wave! Intuitively, we have set the initial velocity of the string in such a way, given by Eq. (5), as to cancel the backward wave.

(ii) Again for an infinite string, suppose that $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ are zero for $|x| > a$, for some real number $a > 0$. Prove that if $t+x > a$ and $t-x > a$, then the displacement $u(x,t)$ of the string is constant. Relate this constant to $g(x)$.

Solution: D'Alembert's solution for the wave equation is

$$u(x,t) = \frac{1}{2}\left(f(x-t) + f(x+t)\right) + \frac{1}{2}\int_{x-t}^{x+t} g(s)\,ds$$

If $x + t > a$ and $t - x > a$ (this is the Region R_4!), then $|x+t| > a$ and $|x-t| > a$, so that $f(x \pm t) = 0$. Furthermore, with $x - t < -a$ and $x + t > a$ we have

$$\int_{x-t}^{x+t} g(s)\,ds = \int_{-a}^{a} g(s)\,ds = \int_{-\infty}^{\infty} g(s)\,ds = c_a$$

Thus c_a is just the area under the curve $g(x)$, and

$$u(x,t) = \frac{c_a}{2}, \qquad x + t > a, \qquad t - x > a.$$

3 Problem 3

Consider a semi-infinite vibrating string. The vertical displacement $u(x,t)$ satisfies

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad x \geq 0, \qquad t \geq 0 \\ u(0,t) &= 0, \qquad t \geq 0 \\ u(x,0) &= f(x), \qquad \frac{\partial u}{\partial t}(x,0) = g(x), \qquad x \geq 0, \end{aligned} \tag{6}$$

The BC at infinity is that $u(x,t)$ must remain bounded as $x \to \infty$.

(a) Show that D'Alembert's formula solves (6) when $f(x)$ and $g(x)$ are extended to be odd functions.

Solution: Let $\hat{f}(x)$ and $\hat{g}(x)$ be the odd extensions of $f(x)$ and $g(x)$, respectively,

$$\hat{f}(x) = \begin{cases} f(x), & x \geq 0 \\ -f(-x), & x < 0 \end{cases}, \qquad \hat{g}(x) = \begin{cases} g(x), & x \geq 0 \\ -g(-x), & x < 0 \end{cases}$$

You can check for yourself that $\hat{f}(x)$ and $\hat{g}(x)$ are odd functions, i.e. $\hat{f}(-x) = -\hat{f}(x)$ and $\hat{g}(-x) = -\hat{g}(x)$. We now write D'Alembert's solution with $\hat{f}(x)$ and $\hat{g}(x)$ replacing $f(x)$ and $g(x)$:

$$u(x,t) = \frac{1}{2}\left(\hat{f}(x-t) + \hat{f}(x+t) + \int_{x-t}^{x+t} \hat{g}(s)\,ds\right) \tag{7}$$

Eq. (7) is D'Alembert's solution for the following wave problem on the infinite string:

$$\begin{aligned} u_{tt} &= u_{xx}, \qquad -\infty < x < \infty, \qquad t \geq 0 \\ u(x,0) &= \hat{f}(x), \qquad \frac{\partial u}{\partial t}(x,0) = \hat{g}(x), \qquad -\infty < x < \infty. \end{aligned}$$

Hence we know (7) satisfies the wave equation, by the way we found D'Alembert's formula. Of course, you can check that directly:

$$\begin{aligned} u_x &= \frac{1}{2}\left(\hat{f}'(x-t) + \hat{f}'(x+t) + \hat{g}(x+t) - \hat{g}(x-t)\right) \\ u_{xx} &= \frac{1}{2}\left(\hat{f}''(x-t) + \hat{f}''(x+t) + \hat{g}'(x+t) - \hat{g}'(x-t)\right) \\ u_t &= \frac{1}{2}\left(\hat{f}'(x-t)(-1) + \hat{f}'(x+t) + \hat{g}(x+t) - \hat{g}(x-t)(-1)\right) \\ u_{tt} &= \frac{1}{2}\left(\hat{f}''(x-t)(-1)^2 + \hat{f}''(x+t) + \hat{g}'(x+t) - \hat{g}'(x-t)(-1)^2\right) \end{aligned}$$

Thus $u_{tt} = u_{xx}$. Also, for $x \geq 0$,

$$\begin{aligned} u(x,0) &= \hat{f}(x) = f(x) \\ u_t(x,0) &= \hat{g}(x) = g(x) \end{aligned}$$

Thus (7) satisfies the ICs. Lastly,

$$u(0,t) = \frac{1}{2}\left(\hat{f}(-t) + \hat{f}(t) + \int_{-t}^{t} \hat{g}(s)\,ds\right)$$

But since $\hat{f}$ is odd, $\hat{f}(-t) = -\hat{f}(t)$ and since $\hat{g}(s)$ is odd, the integral of $\hat{g}(s)$ over a region symmetric about the origin is zero! Hence

$$u(0,t) = \frac{1}{2}\left(-\hat{f}(t) + \hat{f}(t) + 0\right) = 0$$

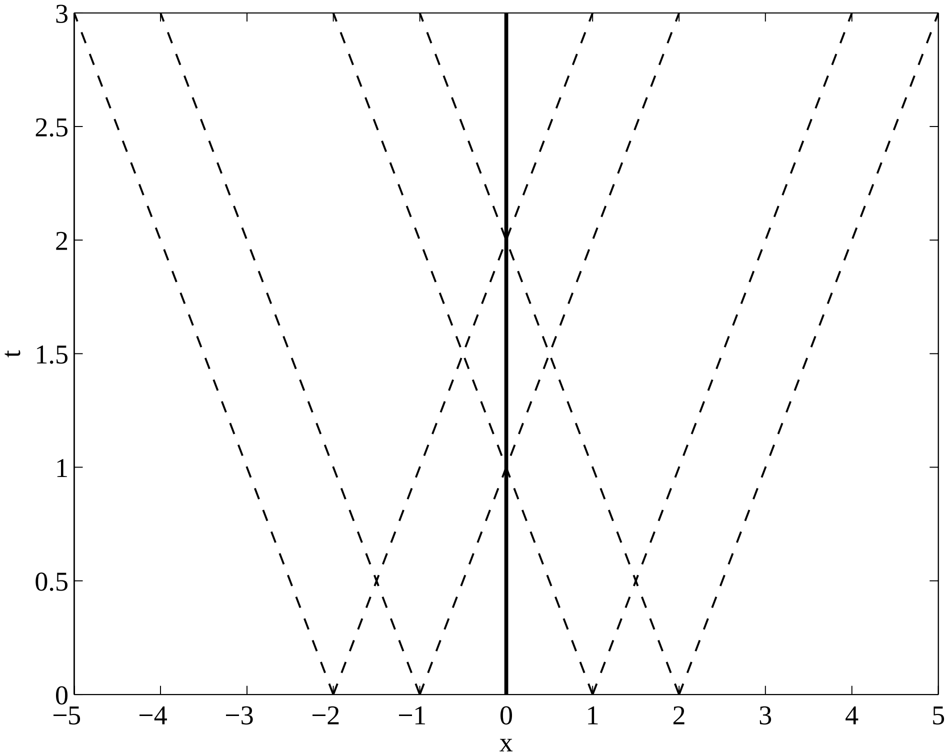

Figure 5: Plot of characteristics for 3(b).

which verifies (7) satisfies the fixed string ($u = 0$) BC at $x = 0$.

(b) Let

$$f(x) = \begin{cases} \sin^2(\pi x), & 1 \le x \le 2 \\ 0, & 0 \le x \le 1, \quad x \ge 2 \end{cases}$$

and $g(x) = 0$ for $x \ge 0$. Sketch u vs. x for $t = 0, 1, 2, 3$.

Solution: D'Alembert's solution reduces to

$$u(x,t) = \frac{1}{2}\left(\hat{f}(x-t) + \hat{f}(x+t)\right)$$

Solving this reduces to finding where $x - t$ and $x + t$ are and whether they are negative. The important characteristics are $x \pm t = \pm 1, \pm 2$. A drawing is useful. The characteristics are plotted in Figure 5 and the solution $u(x, t_0)$ at times $t_0 = 0, 1, 2, 3$ in Figure 6.

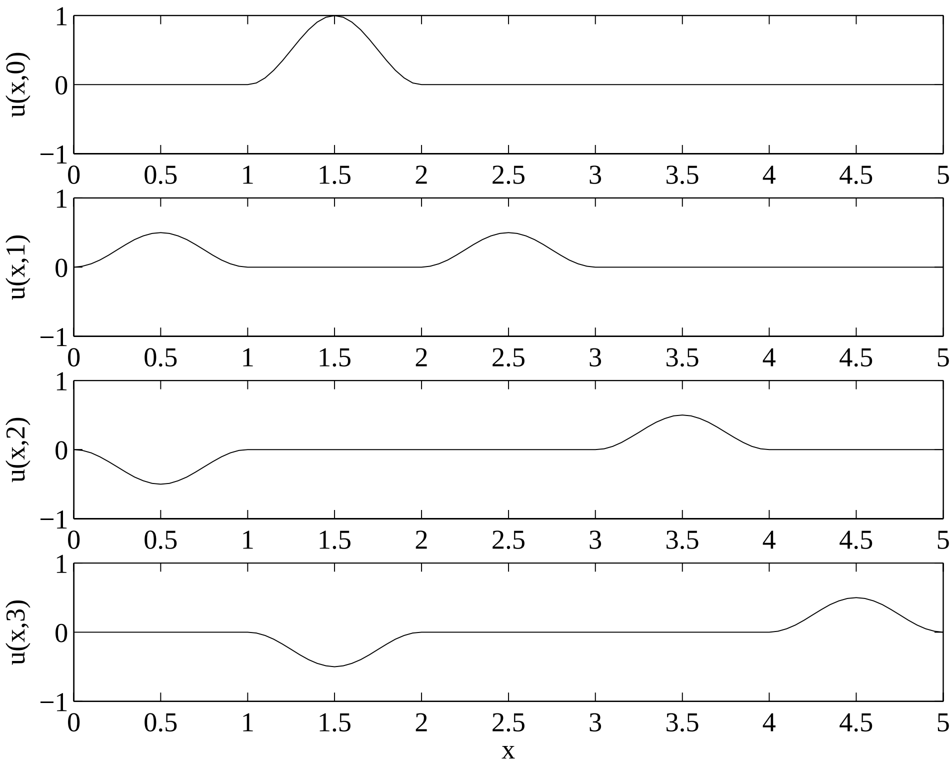

Figure 6: Plot of $u(x, t_0)$ for $t_0 = 0, 1, 2, 3$ for 3(b).

4 Problem 4

The acoustic pressure in an organ pipe obeys the 1-D wave equation (in physical variables)

$$p_{tt} = c^2 p_{xx}$$

where c is the speed of sound in air. Each organ pipe is closed at one end and open at the other. At the closed end, the BC is that $p_x(0, t) = 0$, while at the open end, the BC is $p(l, t) = 0$, where l is the length of the pipe.

(a) Use separation of variables to find the normal modes $p_n(x, t)$.

(b) Give the frequencies of the normal modes and sketch the pressure distribution for the first two modes.

(c) Given initial conditions $p(x, 0) = f(x)$ and $p_t(x, 0) = g(x)$, write down the general initial boundary value problem (PDE, BCs, ICs) for the organ pipe and determine the series solutions.

Solution: Separate variables

$$p_n(x, t) = X(x) T(t)$$

so that the PDE becomes

$$\frac{T''}{c^2 T} = \frac{X''}{X}$$

and since the left side is a function of t only and the right a function of x only, then both sides equal a constant $-\lambda$:

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$$

The boundary conditions are

$$0 = \frac{\partial p}{\partial x}(0,t) = X'(0)\,T(t), \qquad 0 = p(l,t) = X(l)\,T(t)$$

For a non-trivial solution, we must have $X'(0) = 0$ and $X(l) = 0$. We obtain the Sturm Liouville problem

$$X'' + \lambda X = 0; \qquad X'(0) = 0 = X(l)$$

By replacing x with x/l in problem 4 on assignment 1, the eigenfunctions and eigenvalues are

$$X_n(x) = \cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right), \qquad \lambda_n = \frac{(2n-1)^2\pi^2}{4l^2}, \qquad n = 1, 2, 3, ...$$

The corresponding time functions are

$$T_n(t) = \alpha_n \cos\left(c\sqrt{\lambda_n}t\right) + \beta_n \sin\left(c\sqrt{\lambda_n}t\right)$$

Thus the normal modes are

$$\begin{aligned} p_n(x,t) &= X_n(x)\,T_n(t) \\ &= \cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)\left(\alpha_n \cos\left(\frac{2n-1}{2l}\pi ct\right) + \beta_n \sin\left(\frac{2n-1}{2l}\pi ct\right)\right) \\ &= \gamma_n \cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)\cos\left(\frac{2n-1}{2l}\pi ct - \psi_n\right) \end{aligned}$$

where $\gamma_n = \sqrt{\alpha_n^2 + \beta_n^2}$ and $\psi_n = \arctan(\beta_n/\alpha_n)$.

(b) The angular frequency ω_n of the n'th mode is

$$\omega_n = \frac{2n-1}{2l}\pi c$$

and thus the frequency of the n'th mode is

$$f_n = \frac{\omega_n}{2\pi} = \frac{2n-1}{4}\frac{c}{l}$$

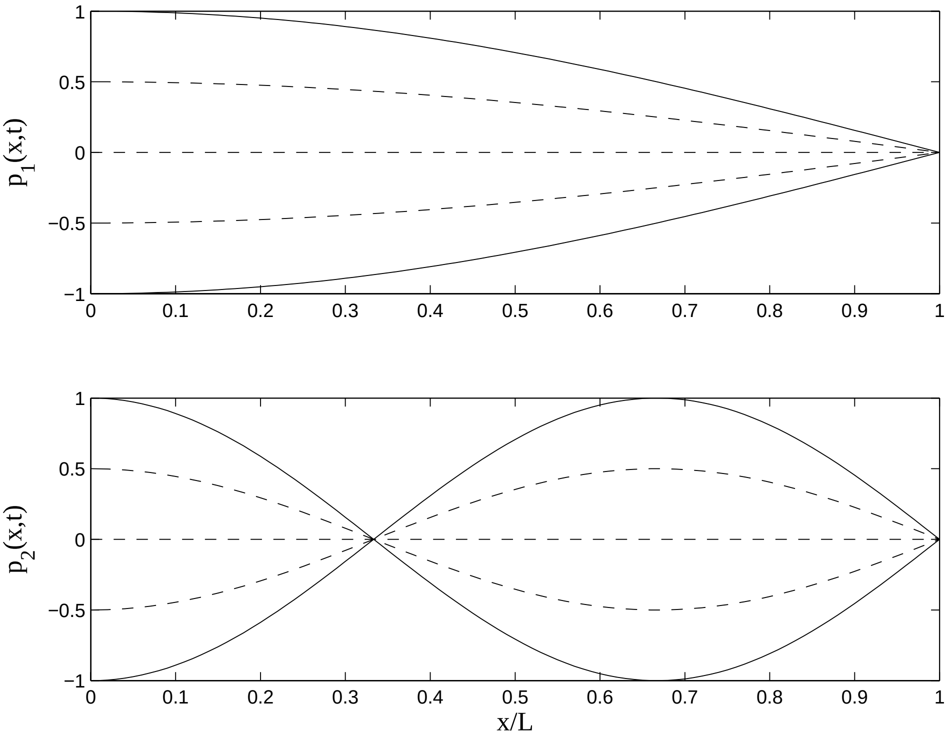

Figure 7: Various phases of the first two normal modes $p_n(x,t)$ $(n = 1, 2)$ with $\gamma_n = 1$. Note that the envelopes (solid lines) are just $\cos((2n-1)\pi x/(2l))$.

Thus, the frequencies and pressure distribution for the first two normal modes ($n = 1, 2$) are

$$\begin{aligned} f_1 &= \frac{1}{4}\frac{c}{l}, \qquad p_1(x,t) = \gamma_1 \cos\left(\frac{\pi}{2}\frac{x}{l}\right)\cos\left(\frac{\pi ct}{2l} - \psi_1\right) \\ f_2 &= \frac{3}{4}\frac{c}{l} = 3f_1, \qquad p_2(x,t) = \gamma_2 \cos\left(\frac{3\pi}{2}\frac{x}{l}\right)\cos\left(\frac{3\pi ct}{2l} - \psi_n\right) \end{aligned}$$

Various phases of the pressure distributions $p_n(x,t)$ of the first two normal modes are plotted in Figure 7, with $\gamma_n = 1$. Notice that $\partial p/\partial x = 0$ at the close end $(x = 0)$ and $p = 0$ at the right end $(x = l)$. This are like the standing waves that appear when you shake a rope at $x = 0$ attached to a wall at $x = l$.

(c) The general initial boundary value problem for the organ pipe is

$$\begin{aligned} p_{tt} &= c^2 p_{xx}, \qquad 0 < x < l, \qquad t > 0 \\ \frac{\partial p}{\partial x}(0,t) &= 0 = p(l,t), \qquad t > 0, \\ p(x,0) &= f(x), \qquad \frac{\partial p}{\partial t}(x,0) = g(x), \qquad 0 < x < l. \end{aligned}$$

Continuing from above, we including all the modes $p_n(x,t)$ in our series solution for

$p(x,t)$,

$$p(x,t)=\sum_{n=1}^{\infty}p_n(x,t)=\sum_{n=1}^{\infty}\cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)\left(\alpha_n\cos\left(\frac{2n-1}{2l}\pi ct\right)+\beta_n\sin\left(\frac{2n-1}{2l}\pi ct\right)\right)$$

Imposing the ICs gives

$$f(x)=p(x,0)=\sum_{n=1}^{\infty}\cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)\alpha_n$$

$$g(x)=\frac{\partial p}{\partial t}(x,0)=\sum_{n=1}^{\infty}\cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)\frac{2n-1}{2l}c\pi\beta_n$$

These are both cosine series. Multiplying each side by $\cos((2m-1)\pi x/(2l))$ and integrating from $x=0$ to $x=l$ and using orthogonality gives

$$\begin{aligned}\alpha_n &= \frac{2}{l}\int_0^l f(x)\cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)dx,\\ \frac{2n-1}{2l}\pi c\beta_n &= \frac{2}{l}\int_0^l g(x)\cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)dx.\end{aligned}$$

Thus

$$\beta_n=\frac{4}{(2n-1)\pi c}\int_0^l g(x)\cos\left(\frac{2n-1}{2}\pi\frac{x}{l}\right)dx.$$