Lecture Eight
Irregular Singular Points of Ordinary Differential Equations

Solutions expanded around an irregular singular point are distinctive in one aspect: they are usually in the form of an exponential function times a Frobenius series. Due to the factor of the exponential function, a solution near an irregular singular point behaves very differently from that near a regular singular point. It may blow up exponentially, or vanish exponentially, or oscillate wildly.

Let us start with the discussion of irregular singular points with the simpler case of the first-order linear and homogeneous equation

$$y' + p(x)y = 0. \tag{6.40}$$

As we know, the solution of (6.40) is

$$y(x) = ce^{-P(x)}, \tag{6.42}$$

where

$$P(x) = \int p(x)dx$$

with c a constant. If $p(x)$ has a pole of order $(k+1)$ at $x = 0$ and hence has the Laurent series expansion

$$p(x) = \frac{b_{k+1}}{x^{k+1}} + \cdots + \frac{b_1}{x} + a_0 + a_1 x^2 + \cdots,$$

find the solution of (6.40).

Answer

We have

$$P(x) = -\frac{b_{k+1}}{kx^k} + \cdots + b_1 \ln x.$$

Thus the solution of (6.40) is of the form

$$\exp\left(\frac{b_{k+1}}{kx^k} + \cdots\right)x^{-b_1}M(x),$$

where $M(x)$ is a Maclaurin series. Note that $x^{-b_1}M(x)$ is a Frobenius series.

If $p(x)$ has a pole of order $(k+1)$ at x_0 instead of at the origin, and if we are interested in the behavior of the solution near x_0, then we make the change of variable

$X = x - x_0$.

The solution near x_0 is given by the one given above with x replaced by X.

We are now ready to discuss the second-order linear homogeneous differential equations (6.38) with an irregular singular point.

Let x_0 be an irregular singular point of (6.38). If

$$(x - x_0)^{k+1}c(x) = c_0 + c_1(x - x_0) + \cdots$$

and

$$(x - x_0)^{2k+2}d(x) = d_0 + d_1(x - x_0) + \cdots,$$

are both convergent Taylor series, and if at least one of c_0 and d_0 are not zero, then x_0 is called an irregular singular point of rank k. If the order of the pole of $c(x)$ and that of $d(x)$ at x_0 are $(k_1 + 1)$ and $(2k_2 + 2)$ respectively, with k_1 not equal to k_2, then the rank k is equal to the greater of k_1 and k_2.

Note that if $k = 0$, then x_0 is a regular singular point of (6.38).

We mention that, if x_0 is an irregular singular point of rank k, where k is a positive integer, then the solutions of (6.38) are of the form

$$y(x) = \exp[F(x)]Y(x), \tag{6.45}$$

where

$$F(x) = \frac{A_k}{(x-x_0)^k} + \frac{A_{k-1}}{(x-x_0)^{k-1}} + \cdots \frac{A_1}{x-x_0}$$

and

$$Y(x) = \sum a_n (x-x_0)^{n+s}, a_0 \neq 0, a_{-1} = a_{-2} = \cdots = 0 \tag{6.46}$$

is a Frobenius series.

As a general remark, if we wish to find the behavior of the solution of (6.38) at very large values of x, we make the change of variable

$$x = \frac{1}{t}.$$

When x is very large, t is very small. Indeed, $x = \infty$ corresponds to $t = 0$. By making an expansion of the solution around $t = 0$ in the way we just outlined, we obtain an expansion around $x = \infty$. Keeping a few terms of this expansion is often a good approximation of the solution at very large values of x.

The change of the independent variable from x to t is done with

$$\frac{d}{dx} = \frac{d}{d(\frac{1}{t})} = -t^2 \frac{d}{dt},$$

and

$$\frac{d^2}{dx^2} = t^2 \frac{d}{dt}\left(t^2 \frac{d}{dt}\right) = t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt}.$$

Thus (6.38) becomes

$$\frac{d^2y}{dt^2} + \frac{2t - c(\frac{1}{t})}{t^2} \frac{dy}{dt} + \frac{d(\frac{1}{t})}{t^4} y = 0. \tag{6.54}$$

We call $x = \infty$ an ordinary point, a regular singular point, or an irregular singular point of rank k of eq. (6.38), respectively, if $t = 0$ is an ordinary point, a regular singular point, or an irregular singular point of rank k of (6.54), respectively. In particular, we see from (6.54) that the infinity is an irregular singular point of rank k if either $d(x)$ blows up like x^{2k-2} or $c(x)$ blows up like x^{k-1} as x goes to infinity.

If the infinity is an irregular singular point of rank k, then the solution of (6.38) for very large x is of the form

$$y(x) = \exp[A_k x^k + A_{k-1} x^{k-1} + \cdots + A_1 x] Y(x), \tag{6.55}$$

where

$$Y(x) = \sum a_n x^{-n-s}, a_0 \neq 0, a_{-1} = a_{-2} = \cdots = 0, \tag{6.56}$$

which is easily deduced from (6.45) and (6.46) by identifying $(x - x_0)$ with t, or $1/x$. Note that the terms in the series (6.56) are in decreasing powers of x, and hence the series is a useful approximation for x large.

Problem for the Reader:

Find the series solution of

$$y'' + x^{-4} y = 0 \tag{6.57}$$

which is useful for very small values of x.

Answer

The point $x = 0$ is an irregular singular point with rank 1. Thus the series solution near the origin is of the form

$$y = \exp\left(\frac{A_1}{x}\right) Y(x), \tag{6.58}$$

with

$$Y(x) = \sum a_n x^{n+s}, a_0 \neq 0, a_{-1} = a_{-2} = \cdots = 0. \tag{6.59}$$

We substitute (6.58) into (6.57). We get

$$\frac{d^2Y}{dx^2} - \frac{2A_1}{x^2}\frac{dY}{dx} + (\frac{2A_1}{x^3} + \frac{A_1^2 + 1}{x^4})Y = 0. \tag{6.60}$$

The value of A_1 is determined by requiring the most divergent term in the coefficient of Y in (6.60) to vanish. There is, in eq. (6.60), a term in the coefficient of Y which is a fourth-order pole at the point of expansion $x_0 = 0$. Thus we require

$$A_1^2 + 1 = 0,$$

or

$$A_1 = \pm i.$$

Let us choose the root

$$A_1 = i.$$

Then (6.60) becomes

$$\left(\frac{d^2}{dx^2} - \frac{2i}{x^2}\frac{d}{dx} + \frac{2i}{x^3}\right)Y = 0.$$

One of the solutions can be obtained by inspection. We get

$$Y(x) = x.$$

This result can also be obtained if we go through the grind to obtain the recurrence formula

$(n+s-1)(n+s-2)a_{n-1} = 2i(n+s-1)a_n.$

From the recurrence formula with $n = 0$, we get

$s = 1.$

Thus the recurrence formula is

$n(n-1)a_{n-1} = 2ina_n.$

Setting $n = 1$, we obtain from the recurrence formula that

$a_1 = 0,$

and hence $a_2 = a_3 = \bullet\bullet\bullet = 0$. Therefore, the series solution for $Y(x)$ terminates after one term. Thus we have obtained the closed form expression of one of the independent solutions of (6.57) as

$$y_1(x) = x\exp\left(\frac{i}{x}\right).$$

Taking the complex conjugate $y_1(x)$, we get the second independent solution of (6.57) as

$$y_2(x) = x\exp\left(-\frac{i}{x}\right).$$

Problem for the Reader:

Find the series solution of the Bessel equation

$$y'' + x^{-1}y' + (1 - p^2/x^2)y = 0 \tag{6.61}$$

which is useful for very large values of x.

Answer

We may read off the rank at infinity from the Bessel equation (6.61) directly, without changing the independent variable from x to t.

As x goes to infinity, the coefficient of y goes to a constant, or x^0. Therefore, according to this coefficient,

$2k - 2 = 0,$

or

$k = 1.$

The coefficient of y' is x^{-1}. Thus, according to this coefficient,

$k - 1 = -1,$

or

$k = 0.$

The rank of the Bessel equation at $x_0 = \infty$ is the greater of the two, and is unity.

Thus we put

$$y(x) = \exp(A_1x)Y(x)$$

where

$$Y(x) = \sum a_n x^{-n-s}, a_0 = 1, a_{-1} = a_{-2} = \cdots = 0.$$

The Bessel equation becomes

$$(D^2 + 2A_1D + x^{-1}D + A_1^2 + 1 + A_1/x - p^2/x^2)Y = 0.$$

We require the sum of the most divergent terms in the coefficient of Y in the equation above vanish. As $x \to \infty$, the coefficient of Y in the equation above is $(A_1^2 + 1)$. Thus we get

$$A_1 = \pm i.$$

This determines the two roots of A_1.

Take the root

$$A_1 = i.$$

Then the equation for Y becomes

$$Y'' + (2i + x^{-1})Y' + (i/x - p^2/x^2)Y = 0.$$

We have

$$(D^2 + x^{-1}D - p^2/x^2)Y(x) = \sum[(n+s)(n+s+1) - (n+s) - p^2]a_n x^{-n-s-2},$$

and

$$(2iD + i/x)Y(x) = \sum[-2i(n+s) + i]a_n x^{-n-s-1}.$$

Thus we have

$$2i(n + s - 1/2)a_n = [(n+s-1)^2 - p^2]a_{n-1}.$$

Setting $n = 0$, we get

$$s = 1/2.$$

For $n \neq 0$, the recurrence formula is

$$a_n = \frac{(n+p-1/2)(n-p-1/2)}{2in}a_{n-1}. \qquad (6.62)$$

Thus we have

$$a_n = \frac{1}{(2i)^n}\frac{\Gamma(n+p+1/2)\Gamma(n-p+1/2)}{n!\Gamma(p+1/2)\Gamma(-p+1/2)}a_0.$$

Therefore, one of the series solution for the Bessel equation for very large values of x is

$$y_1(x) = x^{-1/2}\exp(ix)\sum\frac{1}{(2ix)^n}\frac{\Gamma(n+p+1/2)\Gamma(n-p+1/2)}{\Gamma(p+1/2)\Gamma(-p+1/2)n!}. \qquad \#$$

The other series solution for the Bessel equation for very large values of x is

$$y_2(x) = x^{-1/2}\exp(-ix)\sum_{n=0}^{\infty}\frac{1}{(-2ix)^n}\frac{\Gamma(n+p+1/2)\Gamma(n-p+1/2)}{\Gamma(p+1/2)\Gamma(-p+1/2)n!}. \qquad (6.64)$$

Equations (6.63) and (6.64) give the solutions of the Bessel equation useful when $|\ x\ |$ is large. This is to be compared with the series solution (6.32) useful when $|\ x\ |$ is small. Since $J_p(x)$ is a solution of the Bessel equation, it must be a linear superposition of y_1 and y_2. Thus we have

$J_p(x) = cy_1(x) + c^\star y_2(x)$.

The coefficient c cannot be determined with the present analysis, but we will be able to show in Chapter 8 that

$c = \frac{e^{-i\pi(1/4+p/2)}}{\sqrt{2\pi}}$.

Thus

$J_p(x) \approx \sqrt{\frac{2}{\pi x}}\cos(x - \pi/4 - p\pi/2),\ x >> 1.$

The series (6.63) and (6.64) differ from the series (6.32) in one important aspect: the former are divergent for all values of x! This can be seen from (6.62), which shows that, when n is very large, the ratio a_n/a_{n-1} goes to infinity.

That these series are divergent does not imply that they are not useful for the purpose of approximation. Indeed, they are called asymptotic series and give very good approximations to the solutions when x is large if the number of terms is chosen appropriately.

The classic example of an asymptotic series is

$$I(x) = \int_0^\infty \frac{e^{-t}}{1+tx}dt, x \geqslant 0. \qquad (6.65)$$

We have

$$I(0) = \int_0^\infty e^{-t} dt = 1.$$

Thus we expect

$$I(x) \approx 1,$$

when x is small.

To seek an approximation better than this, we approximate the factor $(1+tx)^{-1}$ in the integrand of $I(t)$ by

$$\frac{1}{1+tx} \approx 1 - tx + t^2x^2 + \cdots + (-tx)^n.$$

Using this approximation for the integrand of $I(x)$, we get

$$I(x) \approx \int_0^\infty e^{-t}[1 - tx + t^2x^2 + \cdots + (-tx)^n]dt$$
$$= 1 - x + 2!x^2 + \cdots + n!(-x)^n. \qquad (6.66)$$

The first term of this series is unity, an approximation we have already obtained. However, we note that the series in (6.66) diverges for all values of x as we let $n \to \infty$.

Nevertheless, (6.66) is a useful formula for approximation, as we shall prove. We have

$$\frac{1}{1+tx} = 1 - tx + t^2x^2 + \bullet\bullet\bullet + (-tx)^n + \frac{(-tx)^{n+1}}{1+tx},$$

which is exact. Thus we have

$$I(x) = \int_0^\infty e^{-t}[1 - tx + t^2x^2 + \cdots + (-tx)^n]dt + R_n, \qquad (6.67)$$

where

$$R_n = \int_0^\infty e^{-t}\frac{(-tx)^{n+1}}{1+tx}dt.$$

Equation (6.67) is exact. This is to be compared with (6.66), which is obtained by dropping R_n.

How big is the term dropped? We have

$$|\ R_n\ | \le \int_0^\infty e^{-t}(tx)^{n+1}dt = x^{n+1}(n+1)!.$$

Let $x = 0.1$. Then the sum of the first five terms of (6.66), which is of the order of unity, differs from the exact value by R_4, which is estimated to be about one tenth of one percent. We may improve this approximation somewhat by taking more terms. But as the series eventually diverges, it is obviously not a great idea to be too enthused and take a large number of terms in (6.66). Where do we stop? Very roughly, we should stop when the upper bound of R_n no longer goes down as we increase n by unity. For $x = 0.1$, this happens when $n = 10$.

If $x = 0.01$, and if we take one hundred terms in the series (6.57), the error is as small as 10^{-42}.

Homework Problems due Wed next week (Oct 13, 04)
Chapter 6, problem8.
Chapter 7, problem 1 and 2.