Chapter 1 The Differential Operator

A. Calculus

Let us first introduce a shorthand notation for the differential operator:

$$D \equiv \frac{d}{dx}. \tag{1.1}$$

We note that

$$De^{mx} = me^{mx}.$$

This means that, when it operates on e^{mx}, the differential operator D is simply equal to the constant m :

$$D \to m .$$

Indeed,

$$D^2 e^{mx} = m^2 e^{mx}.$$

Thus, when it operates on e^{mx},

$$D^2 \to m^2 .$$

Similarly , when it operates on e^{mx},

$$D^n \to m^n, \, n = 2, 3 \cdots .$$

Therefore, when it operates on e^{mx}, the operator $P(D)$ can be replaced by

$$P(D) \to P(m), \tag{1.2}$$

where $P(D)$ is a polynomial of D .

B. Ordinary Differential Equation

Next we give a brief review of ordinary differential equations (ODE).

We may classify an ODE by its order and by whether it is linear or non-linear.

Generally, It is much more difficult to solve a non-linear ODE than a linear ODE in a closed form. And it is much more difficult to solve a high-order ODE than a low-order ODE in a closed form.

Consider the simplest of differential equations

$$\frac{dy}{dx} + p(x)y = q(x). \tag{1.9}$$

In (1.9), $p(x)$ and $q(x)$ are functions of x. Since the highest order derivative in the equation above is the first derivative, the equation is called an ODE of the first order. And since the equation is linear in y ($\frac{dy}{dx}$ is regarded as linear in y) , the equation is called linear. Being linear and of the lowest order possible, eq. (1.9) can be solved in a closed form, no matter how ugly $p(x)$ and $q(x)$ are.

If $q = 0$, the equation is called homogeneous.

If $q \neq 0$, the equation is called inhomogeneous.

The solution of eq. (1.9) with $q(x) \neq 0$ is

$y(x) = e^{-P(x)}(\int_0^x e^{P(x')}q(x')dx' + c).$

where
$P(x) \equiv \int_0^x p(x')dx'.$ (1.13).
The solution has one arbitrary constant.

Problem for the reader:
Find a particular solution of the following differential equation:
$(D^{100} + 1)y = e^x$.

Answer:
There is more than one way to find a particular solution of this equation, but we advocate doing it in the following way. We treat D as if it were a number and get

$$y_P = \frac{1}{D^{100} + 1} e^x,$$

where y_P is a particular solution of the differential equation. By using (1.2) with $m = 1$, we get

$$y_P = \frac{1}{1^{100} + 1} e^x = \frac{1}{2} e^x \quad .$$

Example 2
Find the general solution of
$y^{(100)} + y = \cosh x.$
Answer
Step one: we solve the corresponding homogeneous equation

$$Y^{(100)} + Y = 0 \; .$$

Try

$$Y = e^{mx} \; ,$$

then m must satisfy

$$m^{100} = -1. \qquad (1.19)$$

Equation (1.19) is a polynomial equation which has one hundred roots, which we will denote as m_n , $n = 1, \cdots 100$, with

$$m_n = e^{\dfrac{\pi(2n+1)i}{100}} \ .$$

Thus the complementary solution of the present example is

$$Y = c_1 e^{m_1 x} + \cdots + c_{100} e^{m_{100} x} \ . \tag{1.20}$$

Step two: We find one particular solution of the ODE. We have
$y_P = \dfrac{1}{D^{100}+1} \cosh x = \dfrac{1}{D^{100}+1} \dfrac{1}{2}(e^x + e^{-x})$.

Making use of (1.2), we have

$$y_P = \frac{1}{1+1} \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2} \cosh x \ . \tag{1.21}$$

The most general solution is

$$y = y_P + Y \ . \tag{1.22}$$