

Lecture 4

Second-Order Partial Differential Equations

In this lecture and in the next, we'll briefly review second-order PDEs. We'll begin with one of the simplest of such PDEs: the Laplace equation.

A The Laplace Equation

The Laplace equation appears in many branches of physical sciences, two of which being electrostatics and fluid mechanics.

Let the Laplace equation hold in the upper-half plane in a two dimensional space:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u = 0, y > 0. \quad (4.1)$$

Needless to say, there are (infinitely) many solutions to this equation, corresponding to (infinitely) many different physical situations in the realms of static electricity or incompressible and irrotational fluid flow. In order for the solution of this equation to be unique, we need to impose some conditions on u .

Consider the square in the upper-half plane with the x -axis between $-R$ to R being its lower horizontal boundary. We will think of the upper-half plane as this square in the limit $R \rightarrow \infty$. We require that the value of u is equal to zero at all sides of the square except the lower horizontal side. In the limit $R \rightarrow \infty$, this means that

$$\lim_{r \rightarrow \infty} u(x, y) = 0, \quad (4.2)$$

where

$$r = \sqrt{x^2 + y^2}.$$

In addition, we require that, on the the lower horizontal side of the square which is the x -axis,

$$u(x, 0) = f(x). \quad (4.3)$$

Equations (4.2) and (4.3) constitute a set of boundary conditions for u . More precisely, the values of u on three sides of the square are required to be zero, while those on the fourth side are given by $f(x)$. Equation (4.1) together with these boundary conditions for u is a boundary-value problem.

It may help our perspective if we compare this with the much simpler problem of the Laplace equation in the one-dimensional space, which is

$$\frac{d^2}{dx^2} u(x) = 0.$$

Being of the second order, the equation gives the solution

$$u(x) = ax + b,$$

which has two arbitrary constants. To determine the constants a and b , two conditions on u are required.

We may choose these two conditions to be at the same point of x . For example, we may choose the two conditions to be the value of u and the value of u' at $x = 0$, and the problem is to find the solution for $x > 0$. Such a problem is called an initial-value problem, and the solution is uniquely given by $u = u(0) + u'(0)x$.

Alternately, let the range of x for the equation be $[0, 1]$. Then we may choose these two conditions to be the values of u at the endpoints, say $u(0) = 1$ and $u(1) = 2$, which are conditions imposed at two different points. This is called a boundary-value problem. The solution of this problem is uniquely given by $u = x + 1$.

Let us see what happens if, in a boundary-value problem, we specify the values of u' instead of those of u . We see immediately that there is a difficulty. For, if u is a solution of $u'' = 0$, u' is a constant. Therefore, there is no solution satisfying these boundary conditions unless the two values of u' specified at the endpoints are equal. But there is another difficulty if the specified values of u' at the two endpoints are equal. This is because if u is a solution of this boundary-value problem, so is $u + c$, where c is an arbitrary constant (why?). Thus the solution of such a boundary-value problem is

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not unique.

There are also other kinds of boundary conditions. For example, we may specify the value of u at one of the boundary points, and the value of u' at the other boundary point. As an example, let $u'(0) = 1$ and $u(1) = 0$. Then the solution of the boundary-value problem exists and is unique, and is given by $u(x) = (x - 1)$.

Let us return to the two-dimensional Laplace equation (4.1) with the boundary conditions (4.2) and (4.3), which specify the values of u at the boundary of the upper-half plane. To obtain the solution to this boundary-value problem, we focus on the x -dependence of $u(x, y)$ on x and denote the Fourier integral of u in the x variable by

$$\tilde{u}(k, y) \equiv \int_{-\infty}^{\infty} e^{-ikx} u(x, y) dx. \quad (4.4)$$

This Fourier transform exists because $u(\pm\infty, y) = 0$. Instead of solving for $u(x, y)$, we shall solve for $\tilde{u}(k, y)$. Once $\tilde{u}(k, y)$ is found, we immediately obtain $u(x, y)$ with the use of the Fourier inversion formula.

Problem for the Reader:

Substitute (4.5) into (4.1), obtain an ODE for $\tilde{u}(k, y)$ and solve this equation. Use the boundary conditions to determine $\tilde{u}(k, y)$.

Answer

Since D operating on e^{ikx} is equal to ik , where $D \equiv \frac{\partial}{\partial x}$, we get

$$\left(-k^2 + \frac{\partial^2}{\partial y^2}\right) \tilde{u}(k, y) = 0. \quad (4.6)$$

We see that the PDE becomes an ODE, which is easily solved. This is the advantage of expressing u as a Fourier integral. Solving the ODE, we have

$$\tilde{u}(k, y) = A(k)e^{ky} + B(k)e^{-ky}. \quad (4.7)$$

The boundary value condition (4.2) forbids the solution to blow up as $y \rightarrow \infty$. Thus we may eliminate one of the constants in the solution (4.7). The condition (4.3) enables us to find that

$$u(x, y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx - |k|y} \tilde{f}(k), \quad (4.12)$$

which satisfies both the Laplace equation (4.1) and the boundary conditions (4.2) and (4.3). In the above, $\tilde{f}(k)$ is the Fourier transform of $f(x)$.

The factor $e^{-|k|y}$ provides a convergent factor for the Fourier integral in (4.12) when y is non-zero. Thus the integral in (4.12) converges nicely as one leaves the x -axis, where the values of u are given, into the x - y plane, where y non-zero. The boundary-value problem of the Laplace equation is therefore called well-posed. It is also the kind of problem that people naturally ask themselves, an example being “what is the static electric potential inside Benjamin Franklin’s silver pint can?”

As we recall, for the first-order PDE (2.15), all values of u on a characteristic curve are determined once an initial value on this curve is given. For the Laplace equation, one needs the values of u at all points of the boundary to determine the value $u(x, y)$ at any point in the interior region. Trivially, no value of the solution at any interior point is known if the value of u is given only at one point.

Problem for the Reader:

If $f(x) = \frac{1}{1+x^2}$, find $u(x, y)$.

Answer

$$u(x, y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx - |k|y} \pi e^{-|k|}.$$

The integral above can be easily integrated out and we get

$$u(x, y) = \frac{1+y}{x^2 + (1+y)^2}.$$

We next consider the initial-value problem of the Laplace equation. More precisely, let u satisfying the initial conditions

$$u(x, 0) = f(x),$$

and

$$u_y(x, 0) = g(x).$$

We also require that

$$u(\pm\infty, y) = 0.$$

The last condition ensures that $u(x, y)$, considered as a function of x , has a Fourier transform.

As before, we think of the upper-half plane as an infinite square in the upper-half plane. In the problem we shall treat, two conditions are given on the lower horizontal boundary of the square, one condition is given to each of the two vertical boundary lines, but no condition is imposed on the upper horizontal boundary line. Therefore, this is, more precisely, an initial-value problem with respect to the y variable and a boundary-value problem with respect to the x variable.

Problem for the Reader:

Solve the Laplace equation (4.1) with the initial conditions given above.

Answer

$$u(x, y) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \left[\tilde{f}(k) \cosh ky + \tilde{g}(k) \frac{\sinh ky}{k} \right]. \quad (4.17)$$

Since both $\cosh ky$ and $\sinh ky$ blow up exponentially as a function of k as k goes to infinity, it is impossible for the integral in (4.17) to converge for all values of y unless both $\tilde{f}(k)$ and $\tilde{g}(k)$ vanish more rapidly than an exponential function as $|k|$ goes to infinity.

We call the initial-value problem of the Laplace equation not well-posed.

B The Wave Equation

Consider the wave equation

$$\phi_{xx} - \phi_{tt} = 0. \quad (4.24)$$

The wave equation describes many physical phenomena, a notable example being the propagation of the electromagnetic wave in the vacuum.

Problem for the Reader:

Let

$$u = x + t, v = x - t. \quad (4.25)$$

Express the wave equation (4.24) with the independent variables u and v .

Answer

$$\frac{\partial^2 \phi}{\partial u \partial v} = 0.$$

From the equation above, we see that the solution of the wave equation is a sum of an arbitrary function of u and an arbitrary function of v :

$$\phi(x, t) = f(u) + g(v) = f(x + t) + g(x - t). \quad (4.26)$$

We note from (4.26) that the solution of the second-order wave equation has two arbitrary functions. This is to be compared with that of a first-order PDE of two independent variables, which generally has one arbitrary function. We also note that $f(x + t)$ is a wave travelling to the left with unit velocity, and $g(x - t)$ is a wave travelling to the right with unit velocity.

Problem for the Reader:

Let ϕ satisfy the following initial conditions on the x -axis:

$$\phi(x, 0) = a(x), \phi_t(x, 0) = b(x). \quad (4.27)$$

Find $\phi(x, t)$.

Answer

It is straightforward to show from (4.26) and (4.27) that

$$\phi(x, t) = \frac{a(x + t) + a(x - t)}{2} + \frac{1}{2} \int_{x-t}^{x+t} b(x') dx'. \quad (4.29)$$

Obviously, we may also impose the Cauchy data on other lines or curves, not just the x -axis.

See the textbook for more detailed explanations, more examples, as well as the treatment of other partial differential equations.