

Lectures 16 and 17

Boundary Layers and Singular Perturbation

A Regular Perturbation

In some physical problems, the solution is dependent on a parameter ϵ . When the parameter ϵ is very small, it is natural to expect that the solution not be very different from the one with ϵ set to zero. Consider for example the motion of a harmonic oscillator of unit mass with the spring constant equal to $(1 + \epsilon e^{-t})$. The equation of motion for this problem is

$$\ddot{x} + (1 + \epsilon e^{-t})x = 0, \quad (9.1)$$

with

$$x(0, \epsilon) = 1, \quad \dot{x}(0, \epsilon) = 0, \quad (9.2)$$

where $x(t, \epsilon)$ is the displacement of the harmonic oscillator and \dot{x} is the time derivative of x .

Equation (9.1) can be solved in a closed form and the exact solution is a Bessel function. But if ϵ is very small, the spring constant is approximately equal to unity at all times $t > 0$, and to the lowest-order of approximation, we expect that we may ignore its small deviation from unity. Thus we have

$$x(t, \epsilon) \approx x(t, 0) = \cos t. \quad (9.3)$$

To find the small correction due to the small deviation of the spring constant from unity, one may anticipate that, since the deviation is of the order of ϵ , this small correction is of the order of ϵ as well. Indeed, since all terms in (9.1) are analytic functions of ϵ , we will try to see if there is a solution in the form of a Taylor series of ϵ :

$$x(t, \epsilon) \approx x_0(t) + \epsilon x_1(t) + \cdots + \epsilon^n x_n(t) + \cdots, \quad (9.4)$$

where $x_n(t)$ is independent of ϵ . Substituting (9.4) into (9.1) and (9.2), we get

$$\left(\frac{d^2}{dt^2} + 1 \right) [x_0(t) + \epsilon x_1(t) + \cdots] + \epsilon e^{-t} [x_0(t) + \epsilon x_1(t) + \cdots] = 0, \quad (9.5)$$

$$x_0(0) + \epsilon x_1(0) + \cdots = 1, \quad (9.6)$$

and

$$\dot{x}_0(0) + \epsilon \dot{x}_1(0) + \cdots = 0. \quad (9.7)$$

Setting ϵ in the equations above to zero, we get

$$\ddot{x}_0 + x_0 = 0,$$

with

$$x_0(0) = 1, \quad \text{and} \quad \dot{x}_0(0) = 0.$$

The solution $x_0(t)$ is just $x(t, 0)$ given by (9.3).

Setting to zero the coefficient of ϵ on the leftside of (9.5), we get

$$\ddot{x}_1 + x_1 = -e^{-t} \cos t. \quad (9.8)$$

Similarly, we get from (9.6) and (9.7) that

$$x_1(0) = 0. \quad (9.9)$$

$$\dot{x}_1(0) = 0. \quad (9.10)$$

The solution of (9.8) satisfying (9.9) and (9.10) is

$$x_1(t) = \frac{1}{5} \cos t - \frac{3}{5} \sin t + e^{-t} \frac{2 \sin t - \cos t}{5}.$$

Since both $x_1(t)$ and $x_0(t)$ are $O(1)$ at all times and since ϵ is very small, we have

$$|\epsilon x_1(t)| \ll |x_0(t)|$$

at all times, except when t is near the points where $x_0(t)$ happens to vanish.

One may further obtain the n^{th} -order correction $x_n(t)$ from (9.5)–(9.7). We shall find that $x_n(t)$ is $O(1)$ at all times. Thus

$$|\epsilon^n x_n(t)| \ll |\epsilon^{n-1} x_{n-1}(t)|.$$

The inequalities above indicate that keeping more and more terms in the series of (9.4) makes the approximate solution better and better.

The series (9.4) is a perturbation series, and the method given above in obtaining a perturbation series is called that of regular perturbation.

B Boundary Layer Theory

It may be surprising, but not all problems with a small parameter can be solved by regular perturbation. Let us start the discussion with a simple example.

Consider the solution of the differential equation

$$\epsilon \frac{dy}{dx} + y = 0 \tag{9.11}$$

with the initial condition

$$y(0) = 1.$$

If we set ϵ in the differential equation above to zero, we get

$$y(x) = 0.$$

But this approximate solution does not satisfy the initial condition.

In order to understand why this is so, we solve this differential equation in a closed form. We find that

$$y(x) = e^{-x/\epsilon}.$$

This solution of the differential equation satisfies the initial condition for all values of ϵ not equal to zero. But if we set ϵ to zero, the solution above vanishes, and the initial condition is not satisfied.

Indeed, considered as a function of ϵ , the solution of the differential equation has an essential singularity at $\epsilon = 0$. It is therefore no wonder that it does not have a Maclaurin series expansion.

As a comparison, consider the solution of the equation

$$\frac{dy}{dx} + \epsilon y = 0 \tag{9.12}$$

with the same initial condition. The exact solution is

$$y(x) = e^{-\epsilon x}.$$

The approximate solution obtained by setting ϵ in the differential equation to zero is

$$y(x) = 1$$

which is a good approximation of the exact solution when ϵx is very small.

What makes the method of regular perturbation applicable for one but not for the other? The answer lies in the fact that if we set ϵ to zero, eq. (9.12) remains a first-order differential equation, while eq. (9.11) turns into an algebraic equation. Indeed, while it is always true that the magnitude of ϵy is small compared to that of y , it is not always true that the magnitude of $\epsilon y'$ is small compared to that of y . This is because, if $y(x)$ is a rapidly varying function of x , the magnitude of the derivative of $y(x)$ is much larger than $y(x)$. In such a case, $\epsilon y'(x)$ may be of the same order of magnitude as $y(x)$. Indeed, we may easily verify that this is true for the function $y(x) = e^{-x/\epsilon}$, the solution of (9.11).

We encounter similar difficulties with boundary layer problems in fluid mechanics. Let us consider the second-order differential equation

$$\epsilon y''(x) + a(x)y'(x) + b(x)y(x) = 0, \tag{9.13}$$

which generally cannot be solved in a closed form. We shall choose the interval in which (9.13) holds to be $[0, 1]$. Typically, we require that the solution satisfies the boundary conditions:

$$y(0) = A, y(1) = B. \tag{9.14}$$

The parameter ϵ is very small, while A and B are $O(1)$. Note that the small parameter ϵ is the coefficient of y'' .

This problem is a simplified model of the boundary layer problem in fluid mechanics. As we know, the highest-order term in the Navier-Stokes equation in fluid mechanics is equal to $\nu \nabla^2 \vec{v}$, where \vec{v} is the velocity of the fluid and ν is the kinematic viscosity. The kinematic viscosity is often a small parameter, and the order of the Navier-Stokes equation decreases if we ignore the viscosity.

Equation (9.13) can be solved with good accuracy with the use of the WKB method. (See

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homework problem 1.) However, we gain more insights to this problem by starting afresh.

Let us set $\epsilon = 0$. Then (9.13) becomes

$$a(x)y' + b(x)y = 0.$$

This equation is readily solved and we get

$$y_s(x) = \exp\left[-\int \frac{b(x)}{a(x)} dx\right]. \quad (9.15)$$

The fact is that eq. (9.13) is of the second order, and has two independent solutions. But the approximate equation we obtain is of the first order, and has only one independent solution. Thus the approximate method we use yields only one of the independent solutions. This is to say that we lose one of the independent solutions as we set ϵ in the differential equation to zero. It is impossible to satisfy the two boundary conditions of (9.14) with only one solution.

How do we find the missing solution approximately? To get some insight into the answer, let us go to the much simpler case in which a and b are constants. Then the general solution of (9.13) is equal to

$$c_1 e^{r_1 x} + c_2 e^{r_2 x},$$

where c_1 and c_2 are arbitrary constants and r_1 and r_2 are the roots of the quadratic equation

$$\epsilon r^2 + ar + b = 0.$$

This quadratic equation can be solved easily. But if we set ϵ in this quadratic equation to zero, we get only one root

$$r \approx -b/a,$$

which gives the solution

$$e^{-bx/a}.$$

The solution is equal to the one denoted by $y_s(x)$ in (9.15) with $a(x)$ and $b(x)$ replaced by the constants a and b , respectively. The other root of the quadratic equation is obtained by solving the equation exactly. When ϵ is very small, this root is approximately given by

$$-a/\epsilon.$$

This root tells us that the second solution of the differential equation is approximately

$$e^{-ax/\epsilon}.$$

We see that this solution is a rapidly varying function of x . Indeed, it is a rapidly decreasing function of x if a is positive, and a rapidly increasing function of x if a is negative.

That the missing solution is a rapidly varying function of x also explains why it is not always legitimate to drop the y'' term in (9.13). Let us denote the rapidly varying solution by $y_r(x)$. Then $y_r'(x)$ is of the order of ϵ^{-1} times $y_r(x)$, and $y_r''(x)$ is of the order of ϵ^{-2} times $y_r(x)$. Therefore, the term $\epsilon y_r''(x)$ is bigger than $y_r(x)$ by a large multiple of ϵ^{-1} . Again, while it is true that ϵy , say, is always much smaller than y , it is not always true that $\epsilon y''$ is much smaller than y .

On the other hand, y_s'' and y_s' are both of the same order of magnitude as y_s . Thus it is justified to treat $\epsilon y''$ as a small perturbation term. Thus we write (9.13) as

$$a(x)y_s'(x) + b(x)y_s(x) = -\epsilon y_s''(x), \quad (9.16)$$

and seek the solution $y_s(x)$ in the perturbation series

$$y_s(x) = y_s^{(0)}(x) + \epsilon y_s^{(1)}(x) + \dots$$

Substituting this series into (9.16), we easily find that $y_s^{(0)}(x)$ is given by (9.15), and that

$$a(x) \frac{d}{dx} y_s^{(n)}(x) + b(x) y_s^{(n)}(x) = -\frac{d^2}{dx^2} y_s^{(n-1)}(x), n = 1, 2, \dots \quad (9.17)$$

Thus $y_s^{(n)}(x)$, $n = 1, 2, \dots$, satisfies a first-order ODE with the inhomogeneous term equal to the negative of the second-order derivative of $y_s^{(n-1)}(x)$. Once $y_s^{(0)}$ is found, we may then solve (9.17) with $n = 1$ and obtain $y_s^{(1)}$. Similarly, all $y_s^{(n)}(x)$, $n = 2, 3, \dots$, can be obtained by successive iteration.

We remark that, by treating $\epsilon y_s''$ as a small perturbation, we have implicitly assumed that the solution is slowly varying. Since all terms in the perturbation series so obtained are slowly varying, the assumption is justified and the method of regular perturbation produces successive

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approximations of the slowly varying solution. On the other hand, this method cannot produce the rapidly varying solution. Clearly, if the solution is rapidly varying, it is not justified to treat $\epsilon y''$ as a small perturbation term.

To find successive approximations of the rapidly varying solution, we note that, when $a(x)$ and $b(x)$ are constants, the rapidly varying function is $\exp(-ax/\epsilon)$. Thus we surmise that, when $a(x)$ and $b(x)$ depend on x , the rapidly varying solution is approximately $\exp[-\epsilon^{-1} \int a(x) dx]$. Therefore, we shall put

$$y_r(x) = \exp\left[-\epsilon^{-1} \int a(x) dx\right] v(x). \quad (9.18)$$

With the rapidly varying behavior of $y_r(x)$ taken up by the exponential factor, we expect $v(x)$ to be slowly varying. Substituting (9.18) into (9.13), we get

$$av' + (a' - b)v = \epsilon v''. \quad (9.19)$$

Treating the term $\epsilon v''$ as a small perturbation we may solve (9.19) with regular perturbation. We put

$$v(x) = v_0(x) + \epsilon v_1(x) + \dots, \quad (9.20)$$

and get

$$v_0(x) = \frac{1}{a(x)} \exp\left[\int \frac{b(x)}{a(x)} dx\right] \quad (9.21)$$

as well as

$$a \frac{d}{dx} v_n + (a' - b)v_n = \frac{d^2}{dx^2} v_{n-1}. \quad (9.22)$$

Note that, just like (9.17) for $y_s^{(n)}$, (9.22) is a first-order differential equation for v_n and can be solved in a closed form. Thus we may obtain all v_n by successive iteration. We note that this procedure yields us only the solution which is in the form of (9.18), where v is slowly varying. Hence it does not yield y_s .

If $y_r(x)$ is rapidly decreasing, we expect that it is appreciable only in a small neighborhood near the lower endpoint $x = 0$, and drops to very small values as x gets away from the origin. The graph for the solution (9.23) has a bump near $x = 0$. The solution is said to have a boundary layer near the lower endpoint.

Similarly, if $y_r(x)$ is rapidly increasing, we expect it to be appreciable only near the upper endpoint $x = 1$.

In a specific calculation, we may, instead of plugging in a and b into the formulae given above, obtain both y_{out} and $y_{in}(x)$ directly from (9.13). First of all, the differential equation for y_{out} is obtained by neglecting the term $\epsilon y''$ in (9.13). The resulting equation is of the first-order and can be solved in a closed form. The multiplicative constant in this solution can be fixed with the boundary condition at the endpoint where $y_r(x)$ is negligible. The equation for y_{in} can be obtained by treating the functions a and b as constants.

Problem for the Reader:

Solve approximately the equation

$$\epsilon y'' + (1+x)y' + y = 0, \quad 0 < x < 1,$$

with the boundary conditions

$$y(0) = y(1) = 1.$$

Answer

We have

$$a(x) = 1 + x > 0,$$

thus $y_r(x)$ is rapidly decreasing and there is a boundary layer near $x = 0$ with a width of order ϵ .

For x outside of the boundary layer, we have

$$(1+x)y'_{out} + y_{out} = 0.$$

Thus we get

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$$y_{out}(x) = \frac{c}{1+x}.$$

The constant c is determined from the boundary condition at $x = 1$ and we have

$$y_{out}(x) = \frac{2}{1+x}.$$

In particular,

$$y_{out}(0) = 2.$$

To obtain $y_{in}(x)$ near $x = 0$, we make the approximation

$$a(x) \approx a(0) = 1.$$

Thus

$$\epsilon y_{in}'' + y_{in} \approx 0.$$

Therefore we have

$$y_{in}(x) = 2 + c' e^{-x/\epsilon}.$$

The boundary condition

$$y(0) = 1$$

requires that

$$c' = -1.$$

Thus

$$y_{in}(x) = 2 - e^{-x/\epsilon}.$$

We also find that

$$y_{uniform} = \frac{2}{1+x} - e^{-x/\epsilon}.$$

The solution is plotted in the figure below. We see that it is given by $2/(1+x)$ almost everywhere. But the function $2/(1+x)$ does not satisfy the boundary condition at $x = 0$. Thus the rapidly varying solution y_r kicks in near the point $x = 0$ and the solution y goes through a rapid transition inside the boundary layer.

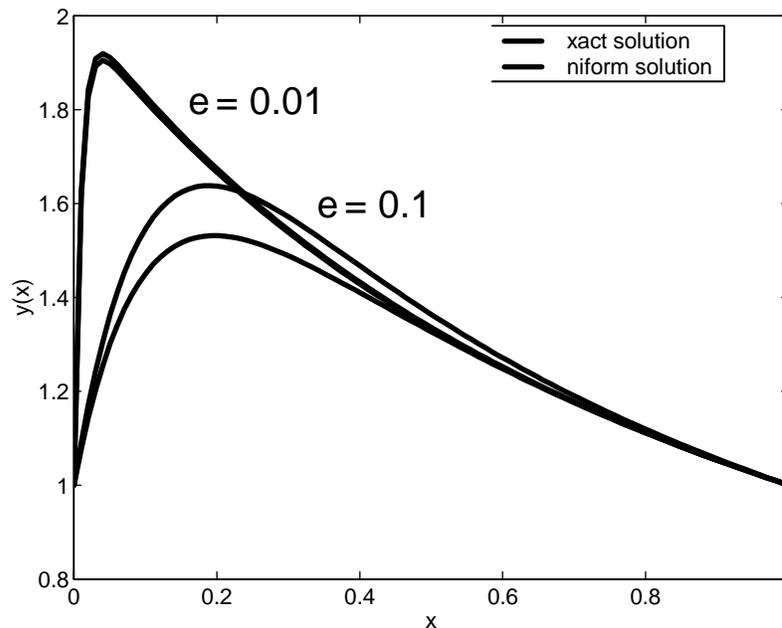


Figure 9.1.

C Turning Points

In the preceding section, we assume that $a(x)$ is of the same sign throughout the interval in which the differential equation (9.13) holds. The solutions $y_r(x)$ and $y_s(x)$, given by (9.24) and (9.15) respectively, are good approximations of the two independent solutions of the differential equation.

But if x_0 is a simple zero of $a(x)$, the function $a(x)$ changes sign at x_0 , and the approximations

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we have made in the preceding section fail in the neighborhood of x_0 . Borrowing the terminology in quantum mechanics, we shall call x_0 a turning point of eq. (9.13).

To obtain approximate solutions near a turning point, we approximate $a(x)$ and $b(x)$ in (9.13) by algebraic functions of x , and solve the resulting equation exactly. This is the same approach as the one used to handle the turning point of the wave equation discussed in Chapter 7. The only difference is that the approximate solutions for (9.13) near a turning point are parabolic cylinder functions rather than Airy functions, as we shall show.

It will be convenient get rid of the y' term in (9.13). Let us write (9.13) as

$$(D + \frac{a}{\epsilon})Dy + \frac{b}{\epsilon}y = 0.$$

This tells us that, to eliminate the y' term, we put

$$y = \exp\left(-\int^x dx' \frac{a(x')}{2\epsilon}\right)Y(x). \quad (9.42)$$

This is because, as we move the exponential factor to the left of D , we make the replacement

$$D \rightarrow D - \frac{a}{2\epsilon}.$$

Thus eq. (9.13) becomes

$$\left(D + \frac{a}{2\epsilon}\right)\left(D - \frac{a}{2\epsilon}\right)Y + \frac{b}{\epsilon}Y = 0,$$

or

$$D^2Y + \left[\frac{2b(x) - a'(x)}{2\epsilon} - \frac{a^2(x)}{4\epsilon^2}\right]Y = 0. \quad (9.43)$$

Since $1/(4\epsilon^2)$ is very large, we see that an alternative way to solve eq. (9.43) is to use the WKB method, approximating $\eta(x)$ in (7.16) by

$$\sqrt{\frac{a^2 + 2\epsilon(a' - 2b)}{4\epsilon^2}} \simeq \frac{|a|}{2\epsilon} + \frac{a' - 2b}{2|a|}.$$

The WKB approximation is valid provided that $a(x)$ does not vanish. This approximation yields for Y two WKB solutions, one of them a rapidly increasing function of x , while the other a rapidly decreasing function of x . To obtain the solutions of (9.13), we multiply these two WKB solutions by the exponential factor of (9.42), getting the slowly varying solution y_s and the rapidly varying solution y_r . (See homework problem 1).

The WKB approximation fails at a turning point. Let x_0 be a turning point. We shall assume that x_0 is a simple zero of $a(x)$. Then we have, near $x = x_0$,

$$a(x) \approx \alpha(x - x_0)$$

and

$$b(x) \approx \beta.$$

By (7.18), the WKB approximation holds only if

$$|x - x_0| \gg \sqrt{\epsilon}.$$

Fortunately, when x is close to x_0 , or more precisely

$$|x - x_0| \ll 1,$$

eq. (9.43) is approximately

$$\frac{d^2Y}{dx^2} + \left[\frac{2\beta - \alpha}{2\epsilon} - \frac{\alpha^2(x - x_0)^2}{4\epsilon^2}\right]Y \approx 0, \quad (9.44)$$

which we will be able to solve in a closed form.

Before we show how to do this, we first note that, when x is near x_0 , we have

$$\exp\left(-\int_{x_0}^x dx' \frac{a(x')}{2\epsilon}\right) \simeq \exp\left(-\frac{\alpha(x - x_0)^2}{4\epsilon}\right).$$

Thus, for x close to x_0 , the solution y is related to Y by

$$y \approx e^{-\alpha(x-x_0)^2/(4\epsilon)}Y. \quad (9.45)$$

We here take note that, if α is positive, the factor multiplying Y in (9.45) drops exponentially as $|x - x_0|$ increases; on the other hand, if α is negative, this factor grows exponentially as $|x - x_0|$ increases.

We shall scale the variable $(x - x_0)$ by

$$x - x_0 = \sqrt{\frac{\epsilon}{|\alpha|}} X. \quad (9.46)$$

Then (9.44) becomes

$$\frac{d^2 Y}{dX^2} + \left[\frac{\beta}{|\alpha|} - \frac{\text{sign}(\alpha)}{2} - \frac{X^2}{4} \right] Y = 0, \quad (9.47)$$

where

$$\text{sign}(\alpha) = \alpha/|\alpha|.$$

We note that, with the independent variable scaled this way, the resulting equation (9.47) is independent of the small parameter ϵ . As a result, the solution of (9.47) varies by an amount $O(1)$ when X varies by an amount $O(1)$. Thus we say that the scale of X is unity. By (9.46), the scale of $(x - x_0)$ is $\sqrt{\epsilon}$. (We have made the implicit assumption that α is of the order of unity.) Therefore, there is a boundary layer with width $\sqrt{\epsilon}$ near the turning point x_0 .

There is a way to see directly from (9.13) that the width of the boundary layer near $x = x_0$ is of the order of $\sqrt{\epsilon}$. Let us choose $x_0 = 0$ for convenience and write eq. (9.13) near the turning point as

$$\epsilon y'' + \alpha xy' + \beta y \approx 0.$$

To determine the scale of x , we make the change of variable

$$x = \delta X.$$

Then we have

$$\frac{\epsilon}{\delta^2} \frac{d^2}{dX^2} y + \alpha X \frac{d}{dX} y + \beta y = 0.$$

We see that if we choose

$$\delta = \sqrt{\epsilon},$$

the differential equation above is independent of the small parameter ϵ . Thus both solutions expressed as functions of X are independent of ϵ . This means that the scale of X for these solutions is unity, or the scale of x for these solutions is $\sqrt{\epsilon}$ as concluded earlier.

Equation (9.47) is a parabolic cylinder equation, which is usually written in the generic form

$$\frac{d^2 \phi}{dz^2} + (v + 1/2 - z^2/4)\phi = 0. \quad (9.48)$$

One of the solutions of eq. (9.48) is the parabolic cylinder function denoted by $D_\nu(z)$. Since equation (9.48) is unchanged with the replacement of $z \rightarrow -z$, $D_\nu(-z)$ also satisfies equation (9.48). In addition, (9.48) is unchanged if we make the replacements $z \rightarrow iz$ as well as $\nu \rightarrow -\nu - 1$. Thus $D_{-\nu-1}(iz)$ and $D_{-\nu-1}(-iz)$ are also solutions of eq. (9.48).

We shall choose the two independent solutions of y near the turning point to be

$$y_1 = e^{-\alpha x^2/(4\epsilon)} D_\nu \left(\sqrt{\frac{|\alpha|}{\epsilon}} x \right), \quad (9.49)$$

and

$$y_2 = e^{-\alpha x^2/(4\epsilon)} D_\nu \left(-\sqrt{\frac{|\alpha|}{\epsilon}} x \right), \quad (9.50)$$

where

$$\nu = \frac{\beta}{|\alpha|} - \frac{\text{sign}(\alpha) + 1}{2}.$$

The asymptotic forms of the parabolic cylinder function are given by

$$\begin{aligned} D_\nu(X) &\approx X^\nu e^{-X^2/4}, & X \rightarrow \infty, \\ &\approx \frac{\sqrt{2\pi}}{\Gamma(-\nu)} |X|^{-\nu-1} e^{X^2/4}, & X \rightarrow -\infty. \end{aligned}$$

We shall replace the parabolic cylinder functions in (9.49) and (9.50) by these asymptotic forms when $|X|$ is large.

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The qualitative behaviors of y_1 and y_2 depend on the sign of α . This is because the exponential factor in (9.49) and (9.50) are either growing exponentially or vanishing exponentially as $|X|$ becomes large, depending on the sign of α . Thus there are two separate cases to discuss.

Case of $\alpha > 0$

For $\alpha > 0$, we have

$$\nu = \frac{\beta}{\alpha} - 1.$$

Also, the exponential function vanishes as $|X|$ increases. Thus we have

$$\begin{aligned} y_1 &\rightarrow (X)^{-1+\beta/\alpha} e^{-X^2/2}, & X \rightarrow \infty, \\ &\rightarrow \frac{\sqrt{2\pi}}{\Gamma(1-\beta/\alpha)} (|X|)^{-\beta/\alpha}, & X \rightarrow -\infty \end{aligned} \quad (9.51)$$

and

$$\begin{aligned} y_2(x) &\rightarrow \frac{\sqrt{2\pi}}{\Gamma(1-\beta/\alpha)} X^{-\beta/\alpha}, & X \rightarrow \infty \\ &\rightarrow |X|^{-1+\beta/\alpha} e^{-X^2/2}, & X \rightarrow -\infty. \end{aligned} \quad (9.52)$$

The behaviors of these two functions are schematically illustrated below:

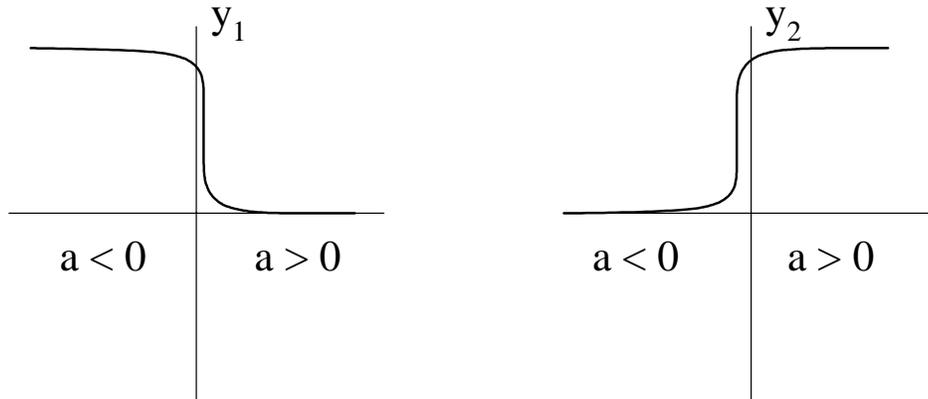


Figure 9.2.

In the region $x > 0$, y_1 is the rapidly varying function, and y_2 is the slowly varying solution, while in the region $x < 0$, y_1 is the slowly varying solution and y_2 is the rapidly varying solution. The role of y_1 interchanges with that of y_2 as one crosses the turning point.

Case of $\alpha < 0$

We have

$$\nu = -\frac{\beta}{\alpha}.$$

Also, the exponential function blows up as $|X|$ becomes large. Thus

$$\begin{aligned} y_1 &\rightarrow X^{-\beta/\alpha}, & X \rightarrow \infty, \\ &\rightarrow \frac{\sqrt{2\pi}}{\Gamma(\beta/\alpha)} |X|^{-1+\beta/\alpha} e^{X^2/2}, & X \rightarrow -\infty, \end{aligned} \quad (9.53)$$

and

$$\begin{aligned} y_2 &\rightarrow \frac{\sqrt{2\pi}}{\Gamma(\beta/\alpha)} X^{-1+\beta/\alpha} e^{X^2/2}, & X \rightarrow \infty, \\ &\rightarrow |X|^{-\beta/\alpha}, & X \rightarrow -\infty. \end{aligned} \quad (9.54)$$

We illustrate in the figure below the behaviors of y_1 and y_2 for $\alpha < 0$.

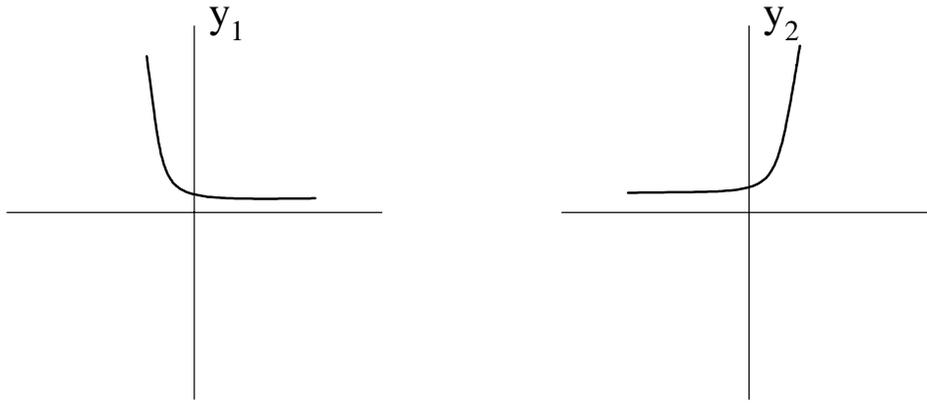


Figure 9.3.

In the region $x > 0$, y_2 is the rapidly varying solution and y_1 the slowly varying solution. And in the region $x < 0$, y_1 is the rapidly varying solution and y_2 is the slowly varying solution.

In the next three sections, we shall show how to match these approximate solutions valid inside the neighborhood of the turning point with the approximate solutions valid outside of the neighborhood of the turning point.

D Turning Point at an Endpoint

Problem for the Reader:

Solve approximately

$$\epsilon y'' + 2xy' + (1 + 3x^3)y = 0, \quad 0 < x < 1, \epsilon \ll 1,$$

with the boundary conditions

$$y(0) = 0, \quad y(1) = 1.$$

Answer

Since $a(x) = 2x \geq 0$, the rapidly varying solution is a decreasing function of x . There is a boundary layer at $x = 0$. Since $x = 0$ is also a turning point, the width of the boundary layer near $x = 0$ is $\sqrt{\epsilon}$.

The solution y_r is negligible outside of the boundary layer. Thus, when $x \gg \sqrt{\epsilon}$, the solution is equal to y_{out} which satisfies

$$2xy'_{out} + (1 + 3x^3)y_{out} = 0.$$

This equation yields

$$y_{out}(x) = \frac{1}{\sqrt{x}} e^{-x^3/2},$$

where we have made use of the boundary condition at $x = 1$.

Inside the boundary layer near $x = 0$, we have

$$y_{in}(x) = e^{-x^2/(2\epsilon)} \left[c_1 D_{-1/2} \left(\sqrt{\frac{2}{\epsilon}} x \right) + c_2 D_{-1/2} \left(-\sqrt{\frac{2}{\epsilon}} x \right) \right].$$

These solutions are good approximations if $x \ll 1$.

There are two arbitrary constants, c_1 and c_2 . We determine one of these constants by matching $y_{in}(x)$ with $y_{out}(x)$ in the overlapping region where both approximations are valid. The parabolic cylinder function solution is valid when $x \ll 1$, while by (7.18), the WKB solutions are valid when

$$\left| \frac{d}{dx} \frac{\epsilon}{x} \right| \ll 1,$$

or

$$x \gg \sqrt{\epsilon}.$$

Thus the overlapping region is

$$\sqrt{\epsilon} \ll x \ll 1.$$

When $x \gg \sqrt{\epsilon}$ we have

$$y_{in}(x) \approx c_2 (2\epsilon)^{1/4} x^{-1/2}.$$

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Also, when $x \ll 1$, we have

$$y_{out}(x) \approx x^{-1/2}.$$

We note that both $y_{in}(x)$ and $y_{out}(x)$ are equal to a constant times $x^{-1/2}$ in the region $1 \gg x \gg \sqrt{\epsilon}$. This is an indication that this region is indeed the overlapping region in which both approximations hold. Joining $y_{in}(x)$ with y_{out} in this region, we get

$$c_2 = (2\epsilon)^{-1/4}.$$

From the boundary condition at $x = 0$, we get

$$c_1 = -c_2.$$

Thus we have

$$y_{in}(x) = e^{-x^2/(2\epsilon)}(2\epsilon)^{-1/4} \left[-D_{-1/2} \left(\sqrt{\frac{2}{\epsilon}} x \right) + D_{-1/2} \left(-\sqrt{\frac{2}{\epsilon}} x \right) \right].$$

As a final observation, we note that $y_{out}(0)$ is infinite. But as we continue $y_{out}(x)$ into the region of the boundary layer, it turns into

$$e^{-x^2/(2\epsilon)}(2\epsilon)^{-1/4} D_{-1/2} \left(-\sqrt{\frac{2}{\epsilon}} x \right).$$

At $x = 0$, the expression above is equal to $(2\epsilon)^{-1/4} D_{-1/2}(0)$, which is a large number but is not infinity.