

Lecture 3

First-Order Partial Differential Equations

Text book: Advanced Analytic Methods in Continuum Mathematics, by Hung Cheng (LuBan Press, 25 West St. 5th Fl., Boston, MA 02111, USA).

In this lecture and the next two lectures, we'll briefly review partial differential equations (PDEs). As PDEs are much more difficult to solve than ODEs, we shall start with the simplest of PDEs, those of the first order.

The good thing about a first-order PDE is this: it can always be “solved” in a closed form. This is true whether the PDE is linear or non-linear, and in the former case, whether it is homogeneous or inhomogeneous.

Having made such a strong statement about the first-order PDE, we must add that the solutions of the first-order PDE are expressed by the solutions of a system of ODE, which may not be exactly solvable. But since ODE is so much simpler to solve than PDE, we consider a PDE “solved” if its solution can be expressed as those of an ODE.

We shall start by making a comparison of the solution of a first-order ODE with those of a first-order PDE.

A Trivial Example

Let us consider the ODE

$$\frac{du(x)}{dx} = 0.$$

The answer is, trivially,

$$u(x) = c,$$

an arbitrary constant. The solution of the ODE is unique if an initial condition is imposed. For example, if we require

$$u(0) = 1,$$

then we have

$$u(x) = 1.$$

In contrast, let us consider the PDE

$$u_x(x, y) = 0, \tag{3.1}$$

where

$$u_x \equiv \frac{\partial u}{\partial x}.$$

The solution of (3.1) is, again trivially,

$$u(x, y) = f(y). \tag{3.2}$$

Let us describe what this obvious result means in some detail. Eq. (3.2) says that $u(x, y)$ is a function of y only, and is independent of x . Therefore, if the value of $u(x, y)$ at an initial point on the y -axis, say $(0, a)$, is equal to b , i.e.,

$$u(0, a) = b,$$

then $u(x, y)$ is equal to b on the entire line $y = a$, i.e.,

$$u(x, a) = b.$$

We call the line $y = a$ a characteristic curve of the PDE (3.1). It is remarkable that u on this characteristic curve is independent of the value of u anywhere else. To find the value of u on this characteristic curve, we only need the value of u at a point on this characteristic curve, just as with

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the case of the first-order ODE discussed above. Clearly, this initial point does not have to be on the y axis.

If the values of $u(x, y)$ on the y axis between $a_1 \leq y \leq a_2$ are given, then the values of $u(x, y)$ are known in the strip of the x - y plane with $a_1 \leq y \leq a_2$.

If the values of $u(x, y)$ are given on a curve, then the values of $u(x, y)$ are known in a strip obtained by sweeping the curve horizontally.

But if the curve intercepts a horizontal line at two points, then there is a problem. For there can be no continuous solution unless the values assigned for u at these two points happen to be equal.

The situation gets worse if the curve on which the initial data is assigned is a horizontal line. In such a case, the initial data is in conflict with the PDE unless all initial values on this horizontal line are equal to the same constant. In such a case, the value of the solution at any point not on the horizontal line remains unknown.

While this example is admittedly trivial, the solution of many first-order PDEs can be obtained by slightly generalizing what we've just done.

B Linear Homogeneous PDEs

Consider the PDE

$$a(x, y)u_x(x, y) + b(x, y)u_y(x, y) = 0, \quad (3.3)$$

which is linear and homogeneous.

Let

$$\Delta u(x, y) \equiv u(x + \Delta x, y + \Delta y) - u(x, y)$$

be the variation of u as the coordinates are varied by $(\Delta x, \Delta y)$. By the Taylor expansion,

$$\Delta u = u_x \Delta x + u_y \Delta y,$$

where the higher order terms, negligible as Δx and Δy become infinitesimal, are ignored.

Question to the Reader:

What values should we choose for Δx and Δy in order for Δu to vanish?

Answer

If u is a solution of (3.3), we have, if $a \neq 0$,

$$u_x = -\frac{b}{a}u_y.$$

Thus

$$\Delta u = \left(-\frac{b}{a}\Delta x + \Delta y\right)u_y.$$

Therefore, Δu vanishes if

$$\Delta y = \frac{b}{a}\Delta x, \quad (3.4a)$$

or

$$\frac{dx}{a(x, y)} = \frac{dy}{b(x, y)}. \quad (3.4b)$$

Therefore, $u(x, y)$ is constant on a curve the slope of which is dictated by (3.4b). We may rewrite (3.4b) as

$$\frac{dy}{dx} = \frac{a(x, y)}{b(x, y)}.$$

This equation does not depend on u . It is self-contained and can be solved on its own, regardless of what u is. We call the solutions of (3.4b) the characteristic curves of the PDE (3.3).

If a vanishes at some points in the region of interest, there is the difficulty of $1/a$ blowing up at these points. However, if $b \neq 0$ at these points, we may express u_y by $-au_x/b$ and get (3.4b).

If both a and b vanish at a point, then the slope of the characteristic curve may be ill-defined at this point. If such is the case, there may be more than one characteristic curves passing through this point.

Problem for the Reader:

Find the characteristic curves of the PDE

$$u_t - u_x = 0.$$

In particular, find the characteristic curve which passes through the point (x_0, t_0) .

Answer

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 The equation for the characteristic curves is

$$\frac{dt}{1} = -\frac{dx}{1},$$

the solutions of which are

$$a = x + t, \tag{3.5}$$

where a is a constant which differs from one characteristic curve to another. In other words, each curve is designated by a value of a . Thus the characteristic curves are a family of curves of one parameter.

Conversely, we may easily find the parameter a associated with the characteristic curve passing through a point (x, t) . For example, the parameter associated with the characteristic curve passing through $(2, 3)$ is, by (3.5),

$$a = 2 + 3 = 5.$$

More generally, the parameter a for the characteristic curve passing through (x_0, t_0) is given by

$$a = x_0 + t_0.$$

Problem for the Reader:

Find the analytic form of the general solution of the PDE

$$u_t - u_x = 0.$$

Answer

Since the values of u are the same on a characteristic curve, its value at (x, t) depends only on the value a associated with the characteristic on which the point (x, t) lies. The value a at the point (x, t) is equal to $(x + t)$. Thus we have

$$u(x, t) = F(x + t) \tag{3.6}$$

where F is an arbitrary function.

Problem for the Reader:

If the initial values of $u(x, t)$ are given by $u(x, 0) = f(x)$, find $u(x, t)$.

Answer

The solution can be expressed in two forms. We may express it in the analytic form by setting $t = 0$ in (3.6). We get

$$f(x) = F(x). \tag{3.7}$$

Thus

$$u(x, t) = f(x + t),$$

which is a wave travelling to the left with the uniform velocity unity.

The solution may also be described graphically by assigning a value of u for each characteristic line in the figure on the right. For example, the value of u on the characteristic passing through the origin is $f(0)$.

The analytic solution is simpler to use for the wave equation given above. Indeed, it is simpler to use for equations in the form of (3.3). But the graphical solutions are preferable for some of the more complicated first-order PDEs not in the form of (3.3).

Problem for the Reader:

Find the characteristic curves of the PDE

$$u_x + (x + y)u_y = 0, \tag{3.8}$$

and draw some of these characteristic curves in the xy -plane.

Answer

The characteristics curves of the PDE (3.8) satisfy

$$\frac{dx}{1} = \frac{dy}{x + y}$$

or

$$\frac{dy}{dx} = x + y.$$

This is a linear and inhomogeneous ODE of the first order with the complementary solution e^x . A

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particular solution is

$$y_p(x) = \frac{1}{D-1}x = -(1 + D + D^2 + \dots)x = -x - 1.$$

Thus the characteristic curves are given by

$$y = ce^x - x - 1, \tag{3.9}$$

or

$$e^{-x}(x + y + 1) = c. \tag{3.10}$$

The characteristic curves for $c = 0, 1, -1$ are drawn in the figure below.

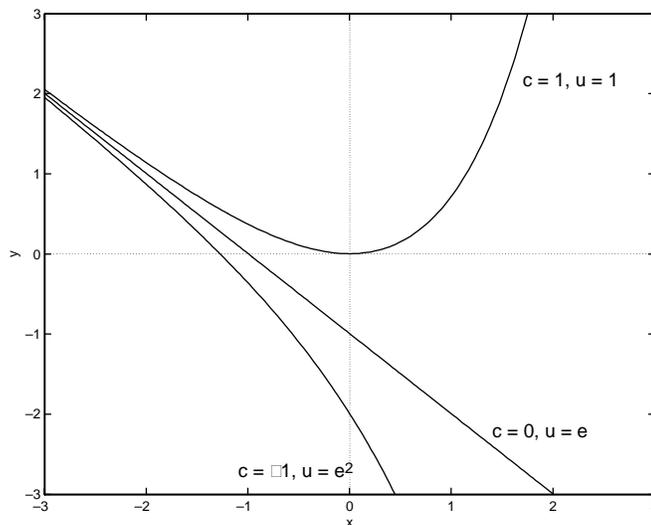


Figure 3.1.

Problem for the Reader:

If $u(x, y)$ of (3.8) satisfies the initial condition

$$u(0, y) = e^{-y}, \tag{3.11}$$

state the values of $u(x, y)$ on the characteristic curves drawn above.

Answer

The coordinates of the intercepts of the characteristic curves (3.10) of $c = 0, 1, -1$ at the y axis are $y = -1, 0, -2$, respectively. According to (3.11), the values of u at these three points are $e, 1, e^2$ as indicated in the figure above.

We may obtain $u(x, y)$ in an analytic form. As the value of u depends only on the characteristic curve it is on, and as each characteristic curve is defined by the parameter c , u depends only on c ; we have

$$u(x, y) = F(c) = F(e^{-x}(x + y + 1)).$$

The function F is determined from the initial condition (3.11). We have

$$e^{-y} = F(y + 1),$$

or

$$F(y) = e^{1-y}.$$

Thus we have

$$u(x, y) = \exp[1 - e^{-x}(x + y + 1)]. \tag{3.12}$$

The solution (3.12) can be viewed geometrically as a surface in the three-dimensional xyu space.

Non-linear first-order ODE can also be solved in a closed form. These and numerous other examples are discussed in the textbook.