

## 18.310 Homework 3

Due September 27th at 6PM

**Instructions:** Remember to submit a separate PDF for each question. Do not forget to include a list of your collaborators or to state that you worked on your own.

1. **Writing Assignment:** Following the discussion in recitation, revise Section 3 (and Section 3 **only**) of the lecture notes on Chernoff Bound. You will find a  $\text{\LaTeX}$  version of the notes in the webpage for Homework 3 under the name “chernoff.tex”.

You have complete editorial freedom and should feel free to add, delete and move text within Section 3, following the instructions given at the top of “chernoff.tex”. You should pay particular attention to instances of missing guiding or explanatory text, such as those that were pointed out in recitation.

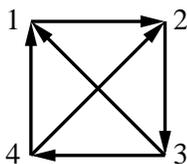
After revising the notes, write a short supporting text (1-2 paragraphs) explaining your most significant revisions to the notes and the rationale behind them.

Please submit online the following documents: i-ii) pdf and latex of your revised lecture notes, iii) pdf of your supporting text (as a separate document).

2. A *tournament* on  $n$  vertices is an orientation of a complete graph on  $n$  vertices, i.e. for any two vertices  $u$  and  $v$ , exactly one of the directed edges  $(u, v)$  or  $(v, u)$  is present. A Hamiltonian path in a tournament is a directed path passing through all vertices exactly once. For example, in the example below,  $1 - 2 - 3 - 4$  is a Hamiltonian path since all edges  $(1, 2)$ ,  $(2, 3)$  and  $(3, 4)$  are directed in the direction of traversal and it visits every vertex once, while for example  $1 - 2 - 4 - 3$  is not a directed path since the second and third edges are directed in the wrong direction.

Show that, for any  $n$ , there exists a tournament with at least  $\frac{n!}{2^{n-1}}$  Hamiltonian paths.

For example for  $n = 4$ , this shows that there exists a tournament on 4 vertices with at least 3 Hamiltonian paths. Here is such a tournament with Hamiltonian paths  $1 - 2 - 3 - 4$ ,  $2 - 3 - 4 - 1$ ,  $3 - 4 - 1 - 2$ ,  $4 - 1 - 2 - 3$ ,  $4 - 2 - 3 - 1$ .



(Hint: think probabilistically...)

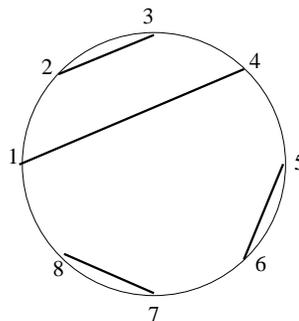
3. The examples of Chernoff bounds discussed in class bound the probability of deviating from the expectation by a certain amount. We can also ask the reverse question of finding intervals where we have high confidence that the random variable will fall into.

Consider flipping a fair coin 100 times. Using the Chernoff bounds shown in class, find  $x$  such that with at least 95% probability we obtain at least  $x$  heads.

4. Recall the sequential choice problem discussed in class:  $n$  objects are shuffled in random order, and are presented to you one at a time. For each object, you know how it compares against all previous ones that you've seen. However, we're now considering the setting in which, rather than your choice being final, you can alter it at any time. Of course, now the strategy becomes obvious: you will switch to the current object if it is better than everything you have seen so far (or equivalently better than the object you last selected). Find the expected number of times you will select a new object.

Formally, given a random permutation of  $[n]$ , find the expected number of entries smaller than all entries before it, and express the answer in terms of  $n$ .

5. Consider  $2n$  points on the plane labelled  $1, 2, \dots, 2n$ , all spaced equally on a circle. A *matching* of these points is a collection of  $n$  straight line segments, with every point being the endpoint of precisely one of the line segments. A matching is *noncrossing* if no two of its line segments cross. Here is an example of a noncrossing matching on 8 points (so  $n = 4$ ).



Determine (with proof) the number of noncrossing matchings of  $2n$  points, as a function of  $n$ . (You might want to look for an appropriate bijection.) Check your result for  $n = 1, 2$  and  $3$ .

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