18.310 Exam 1 practice questions

Collection of problems from past quizzes and other sources. It does not necessarily reflect what will be on the exam on Friday.

1. Suppose one has two coins C_1 and C_2. Coin C_1 gives head with probability 1/2, while coin C_2 gives head with probability 1/3. We pick one of the coins uniformly at random and toss it twice. We get twice the same result. Compute the probability the it was coin C_1 being used.

2. You have a biased coin that comes up heads with probability 1/3. Show that the probability of obtaining 80 heads or more from 90 throws is not more than 0.16.

3. In a permutation of n elements, a pair (j, i) is called an inversion if and only if $i < j$ and i comes after j. For example, the permutation 31542 in the case $n = 5$ has five inversions: (3,1), (3,2), (5,4), (5,2), and (4,2). What is the expected number of inversions in a uniform random permutation of the numbers $1, 2, \cdots, n$?

4. Prove that if C is any subset of $\{100, 101, \ldots, 199\}$ with $|C| = 51$, then C contains two consecutive integers.

5. You have an $n \times 3$ strip of unit squares. Let a_n be the number of ways you can tile it with 1×1, and 3×3 squares. Find the generating function $A(x) = \sum_{n \geq 0} a_n x^n$.

6. Consider the generating function

   $$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

   Suppose

   $$f(x) = \frac{2 + 2x}{1 - 2x - x^2}.$$

   Give an expression for a_n. Can you give a recursion for a_n and initial conditions that would give rise to this generating function?

7. Explain why the following are not valid bijections from $[0 \ldots n-1]$ to $[0 \ldots n-1]$

   1. $f(x) = x + 1$
   2. $f(x) = 2x \mod n$
   3. $f(x) = n - 1 - x$
   4. $f(x) = x^2$

8. Consider a game with three gates, one of which (chosen uniformly) has a prize hidden behind. The player picks one gate, after which one of the other two gates that does not contain the prize is revealed. The player can then choose to change or stay with their choice (of course, it does not make sense to pick the opened gate). Compute the probability of winning if:
   - The player never switches.
   - The player always switches.
9. A chess board is an eight-by-eight square grid, and a rook is a piece that can attack anything in the same row or column. Compute the number of ways of placing six rooks on a eight-by-eight chess board such that no two rooks attack each other.
10. Give a (short) expression for the number of balanced bracket sequences of length $2n$ that starts with (((for all $n \geq 3$.