Course 18.312: Algebraic Combinatorics

Lecture Notes # 10 Addendum by Gregg Musiker
(Based on Lauren Williams' Notes for Math 192 at Harvard)

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1 Introduction to Partitions

A **partition** of n is an ordered set of positive integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $\sum_i \lambda_i = n$ and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

Let $P(n)$ denote the set of all partitions of n with $p(n) = |P(n)|$ and $p(0) = 1$. For example, the partitions of 4 are $\{[4], [3,1], [2,2], [2,1,1], \text{ and } [1,1,1,1]\}$, so $p(4) = 5$.

We have the following the identity of infinite products:

$$\begin{aligned}\prod_{k\geq 1} \frac{1}{1-x^k} &= \prod_{k\geq 1}(1 + x^k + x^{2k} + x^{3k} + \ldots) \\ &= (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)(1 + x^3 + x^6 + \ldots)\cdots\end{aligned}$$

The coefficient of x^n in this infinite product is $p(n)$ since the term of $(1 + x^k + x^{2k} + x^{3k} + \ldots)$ that we pick in each factor determines how many times the number k appears as one of the λ_i's in partition λ. Therefore,

$$\prod_{k\geq 1} \frac{1}{1-x^k} = \sum_{n\geq 0} p(n)x^n. \tag{1}$$

2 Partitions with Odd or Distinct Parts

As an application, we can also leave out some of the j's from the index set for this product, thereby obtaining the generating function for the number of partitions not containing such j's.

For example, if we let $p_o(n)$ denote the number of partitions with only odd parts, we get

$$\prod_{i\geq 1} \frac{1}{1-x^{2i-1}} = \sum_{n\geq 0} p_o(n)x^n. \tag{2}$$

Similarly, if we truncate the factor $(1+x^k+x^{2k}+\dots)$, we can enumerate partitions where k only appears with specified frequencies.

For example, if we let $p_d(n)$ denote the number of partitions where all the λ_i's are distinct, then

$$\prod_{k\geq 1}(1+x^k) = \sum_{n\geq 0} p_d(n)x^n. \tag{3}$$

However, there is an algebraic identity:

$$\begin{aligned} \prod_{k\geq 1}\frac{1}{1-x^{2k-1}} &= \prod_{k\geq 1}\frac{(1-x^{2k})}{(1-x^k)} \\ &= \prod_{k\geq 1}(1+x^k). \end{aligned} \tag{4}$$

Putting together identities (2), (3), and (4), we obtain a Theorem due to Euler, namely the result that $p_o(n) = p_d(n)$.

3 A Combinatorial Proof of $p_o(n) = p_d(n)$

We now describe a combinatorial proof of Euler's Theorem. Let $P_o(n)$ and $P_d(n)$ denote the sets of partitions of n which have odd or distinct parts, respectively. We wish to find a bijection between $P_o(n)$ and $P_d(n)$.

Idea: Let $\lambda \in P_o(n)$ and for all odd k, let n_k be the number of times k appears as a part of λ, i.e. $n_i = \#\{i : \lambda_i = k\}$. Since λ is a partition of a finite number, $n_k = 0$ with a finite set of exceptions.

We write each of the n_i's in binary: $n_i = 2^{m_{i,1}} + 2^{m_{i,2}} + \cdots + 2^{m_{i,r_i}}$ where m_{i,j_1} is different from m_{i,j_2} for each $j_1 \neq j_2$.

We form a new partition λ', defined as the rearrangement of

$$[2^{m_{1,1}}\lambda_1, 2^{m_{1,2}}\lambda_1, \dots, 2^{m_{1,r_1}}\lambda_1, 2^{m_{2,1}}\lambda_2, 2^{m_{2,2}}\lambda_2, \dots].$$

We claim that in general λ' (a) is a partition of $n = |\lambda|$, and (b) has distinct parts.

(a) If we sum the parts of λ', we can reorder and group the summands so that they correspond to the products $n_i\lambda_i$ with n_i written in binary. Thus the parts of λ' sum to the same number as the parts of λ.

(b) Since the λ_i's are all odd, each expression $2^{m_{i,j}}\lambda_i$ is the unique way of writing a certain integer after dividing through by the highest power of two.

We have thus shown a mapping from $P_o(n)$ into $P_d(n)$. To show that this map is a bijection, we construct the inverse map: If $n = \mu_1 + \mu_2 + \cdots + \mu_s$ is partition, using distinct parts, collect all μ_i's with the same highest power of 2 and write down the odd parts with the appropriate multiplicity.

Example: If $\lambda = [7^9, 5^5, 1^6] = [7^{8+1}, 5^{4+1}, 3^{8+4+2+1}, 1^{4+2}]$, then

$$\begin{aligned} \lambda' &= [8\cdot 7, 1\cdot 7, 4\cdot 5, 1\cdot 5, 8\cdot 3, 4\cdot 3, 2\cdot 3, 1\cdot 3, 4\cdot 1, 2\cdot 1] \\ &= [56, 24, 20, 12, 7, 6, 5, 4, 3, 2]. \end{aligned}$$

Exercise 1: Prove algebraically and combinatorially that the number of partitions with no part divisible by k is equal to the number of partitions with no part appearing k times.

4 Euler's Pentagonal Theorem

We now investigate the infinite product whose reciprocal is the generating function for the number of partitions of size n, namely:

$$\prod_{k\geq 1}(1-x^k) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + x^{22} + x^{26} - x^{35} - x^{40} \pm \ldots$$

Observations:

1) All coefficients lie in $\{-1, 0, 1\}$ and the signs satisfy a simple periodic behavior.

2) The exponent sequence $0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, \ldots$ can be split into two subsequences, the second of which consists of the pentagonal numbers $1, 5, 12, 22, 35, \ldots$ described by the formula $f(j) = \frac{3j^2-j}{2}$.

This sequence has such a name because if one draws a regular pentagon where each side has precisely $j-1$ dots, than the entire pentagon consists of $\frac{3j^2-j}{2}$ dots. This motivates

$$\textbf{Euler's Pentagonal Theorem}: \prod_{k\geq 1}(1-x^k) = 1 + \sum_{j\geq 1}(-1)^j\left(x^{\frac{3j^2-j}{2}} + x^{\frac{3j^2+j}{2}}\right).$$

As an application, we can inductively compute $p(n)$. For example,

$$\begin{aligned} p(6) &= p(5) + p(4) - p(1) = 7 + 5 - 1 = 11, \text{ and} \\ p(7) &= p(6) + p(5) - p(2) - p(0) = 11 + 7 - 2 - 1 = 15 \end{aligned}$$

Exercise 2: Use Euler's Pentagonal Theorem to calculate $p(8), p(9)$, and $p(10)$.

5 Proof of Euler's Theorem

We use the fact that $\frac{1}{\prod_{k\geq 1}(1-x^k)} = \sum_{n\geq 0} p(n)x^n$. Thus if we write $\sum_{k\geq 1}(1-x^k) = \sum_{n\geq 0} c(n)x^n$, then

$$(\sum_{n\geq 0} c(n)x^n)(\sum_{n\geq 0} p(n)x^n) = 1.$$

Comparing coefficients, we find that the sequence of $c(n)$'s must satisfy $c(0) = 1$ and $\sum_{k=0}^{n} c(k)p(n-k) = 0$ for all $n \geq 1$. This initial condition and recurrence uniquely determines the sequence of $c(n)$'s.

Consolidating terms, we can write the right-hand-side of Euler's Pentagonal Theorem as $\sum_{j=-\infty}^{\infty}(-1)^j x^{\frac{3j^2+j}{2}}$, so it suffices to show that

$$c(k) = \begin{cases} 1 & \text{if } k = \frac{3j^2+j}{2} \text{ and } j \text{ is even} \\ -1 & \text{if } k = \frac{3j^2+j}{2} \text{ and } j \text{ is odd} \\ 0 & \text{otherwise} \end{cases}.$$

If we let $b(j) = \frac{3j^2+j}{2}$ for all $j \in \mathbb{Z}$, then we wish to show for all n that

$$\sum_{j \text{ even and } b(j)\leq n} p(n-b(j)) - \sum_{j \text{ odd and } b(j)\leq n} p(n-b(j)) = 0,$$

which we rewrite as

$$\sum_{j \text{ even and } b(j)\leq n} p(n-b(j)) = \sum_{j \text{ odd and } b(j)\leq n} p(n-b(j)).$$

We thus want a bijection

$$\phi : \bigcup_{j \text{ even}} P(n-b(j)) \to \bigcup_{j \text{ odd}} P(n-b(j)).$$

We present such a bijection as constructed by Bressoud-Zeilberger. This bijection is actually an involution, $\phi(\phi(\lambda)) = \lambda$ so ϕ is its own inverse. For $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_t] \in P(n-b(j))$ (we remind the reader that $\lambda_1 \geq \lambda_i$ for all i), we set

$$\phi(\lambda) = \begin{cases} [(t+3j-1), (\lambda_1 - 1), (\lambda_2 - 1) \ldots, (\lambda_t - 1)] & \text{if } t+3j \geq \lambda_1 \\ [(\lambda_2+1), \ldots, (\lambda_t+1), 1, 1, \ldots, 1] & \text{if } t+3j < \lambda_1 \end{cases},$$

where there are $\lambda_1 - t - 3j - 1$ copies of 1 in the second case.

Notice that if $t + 3j \geq \lambda_1$, then $\phi(\lambda)$ is a partition of

$$\begin{aligned} & t + 3j - 1 + (\lambda_1 - 1) + (\lambda_2 - 1) + \cdots + (\lambda_t - 1) \\ = \; & t + 3j - 1 + \sum_i \lambda_i - t = 3j - 1 + \sum_i \lambda_i \\ = \; & n - b(j) + 3j - 1 = n - \frac{3j^2 + j}{2} + (3j - 1) \\ = \; & n + \frac{-3j^2 + 5j - 2}{2} = n - b(j-1). \end{aligned}$$

By similar logic, if $t + 3j < \lambda_1$, we see that $\phi(\lambda)$ is a partition of $n - b(j+1)$. Thus ϕ maps elements of $P(n - b(j))$ to an element of $P(n - b(j \pm 1))$. By inspection, we see that $\phi^2 =$ identity, and we conclude that ϕ is the desired bijection.

Example: We calculate $\phi([4, 2, 1]) = [8, 3, 1]$ for $n = 14$ and $j = 2$, which is a partition of $12 = n - b(1)$.

Exercise 3: Calculate $\phi([3, 3, 2, 1, 1, 1])$ for $n = 37$ and $j = 4$.

Solution on next page:

The solution is $[17, 2, 2, 1]$, a partition of $n - b(3) = 22$.

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