Course 18.312: Algebraic Combinatorics

Lecture Notes # 23-24 Addendum by Gregg Musiker

April 6th - 8th, 2009

The following is an outline of the material covered April 6th and 8th in class. This material can be found in Chapter 5 of Stanley's Enumerative Combinatorics Volume 2. Proofs of most of the results are in class notes.

1 Exponential Generating Functions

Definition. Given $f, g : \mathbb{N} \to \mathbb{Z}$, which we think of as counting objects of *sizes* k in two set $\mathcal{F}$ and $\mathcal{G}$, respectively, we define a new function $h : \mathbb{N} \to \mathbb{Z}$ by the following:

$$h(\#X) = \sum_{(S,T)} f(\#S)g(\#T)$$

where X is a finite set and (S, T) disjointly partition X, i.e. $S \cap T = \emptyset$ and $S \cup T = X$. Sets S and T are allowed to be empty.

Definition. We define the **exponential generating function** of sequence $\{f(n)\}$ to be

$$E_f(x) := \sum_{n \geq 0} f(n)\frac{x^n}{n!}.$$

Proposition.

$$E_h(x) = E_f(x)E_g(x).$$

The following is Corollary 5.1.6 of Stanley's Enumerative Combinatorics 2.

Theorem. (The Exponential Formula) Given $f : \{1, 2, \dots\} \to \mathbb{Z}$, define a new function $h : \mathbb{N} \to \mathbb{Z}$ by $h(0) = 1$ and

$$h(\#S) = \sum_{k \geq 1} \sum_{B_1,\dots,B_k} f(\#B_1)f(\#B_2)\cdots f(\#B_k)$$

for $\#S \geq 1$. Here, the sum is over partitions of S, i.e. $B_i \cap B_j = \emptyset$ for all $i \neq j$. We assume these blocks B_i are non-empty, and $B_1 \cup B_2 \cup \cdots \cup B_k = S$. Then

$$E_h(x) = \exp(E_f(x)).$$

The following is Corollary 5.1.8 of Stanley's Enumerative Combinatorics 2.

Theorem. (Permutation Version of the Exponential Formula) Given $f : \{1, 2, \ldots\} \to \mathbb{Z}$, define a new function $h : \mathbb{N} \to \mathbb{Z}$ by $h(0) = 1$ and let $n = \#S$,

$$h(n) = \sum_{\pi \in S_n} f(\#C_1)f(\#C_2)\cdots f(\#C_k)$$

for $\#S \geq 1$. Here, the C_i's are the cycles, thought of as sets of S, in the disjoint cycle decmposition of π. Then

$$E_h(x) = \exp(\sum_{n \geq 1} f(n)\frac{x^n}{n}).$$

Application: The nubmer of simple graphs on n vertices is $2^{\binom{n}{2}}$ and we let $c(n)$ be the number of connected graphs on n vertices.

$$\exp(\sum_{n \geq 1} c(n)\frac{x^n}{n!}) = \sum_{n \geq 0} 2^{\binom{n}{2}}\frac{x^n}{n!}.$$

2 Tree Enumeration

A **tree** is an undirected graph with no cycles. A tree is **rooted** if it has a distinguished vertex (called the root).

Let $T_n = \#$ labeled trees on n vertices.

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A **forest** is a disjoint union of trees. A **rooted forest** is a collection of rooted trees, one root for each tree.

Let $f_n = \#$ of rooted labeled forests on n vertices.

Claim: $T_{n+1} = f_n$ and $t_n = nT_n$.

Bijective Proofs: Peel off root, labeled $(n+1)$ of a rooted tree and left with a rooted forest. A rooted tree is a choice of a labeled tree plus a choice of a vertex to be the root.

A Rooted Forest is a collection of rooted trees, so we can use the exponential formula to count. Let

$$y = E_t(x) = \sum_{n\geq 1} t_n \frac{x^n}{n!} \quad \text{and} \quad E_f(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}.$$

$E_f(x) = \exp(y)$. On the other hand, $t_{n+1} = (n+1)f_n$, so

$$xE_f(x) = \sum_{n\geq 0} f_n \frac{x^{n+1}}{n!} = \sum_{n\geq 0} t_{n+1} \frac{x^{n+1}}{(n+1)!} = E_t(x) = y.$$

Thus $y = E_t(x)$ satisfies $xe^y = y$. We can solve this identity in a way that allows us to compute coefficients of y using a technique known as the **Lagrange Inversion Formula**.

But first, we compute t_n's combinatorially:

Claim. There are $\binom{n}{d_1,d_2,\ldots,d_n} = \frac{(n-1)!}{d_1!d_2!\cdots d_n!}$ rooted trees on $\{1,2,\ldots,n\}$ in which vertex i has outdegree d_i, where the outdegree of a vertex v_i is the number of its neighbors further away from the root. These neighbors are called **children** and the unique neighbor closer to the root is called a **parent**. A vertex with no children is called a **leaf**. (Notice that $\sum_{i=1}^n d_i = n-1$.)

We prove this claim using the **Prüfer code**. Start with a rooted labeled tree T.

1. Locate the leaf with the smallest label.
2. Write down the label of its unique parent. Delete this leaf and its adjoing edges.
3. Go to step 1.

Application: The Prüfer code gives bijections between desired set of sequences and rooted trees with specified outdegrees.

Corollary. $t(n) = n^{n-1}$, the number of sequences of length $(n-1)$ on n letters.

Corollary (Cayley's Theorem). $T(n) = n^{n-2}$, the number of (unrooted) labeled trees on n vertices.

Remark. The Catalan numbers count binary trees in several different ways.

3 Statement of Lagrange Inversion

Given a formal power series $f(x) = a_1 x + a_2 x^2 + a_3 x^3 + \dots$, we say that $f(x)$ has a compositional inverse $f^{\langle -1 \rangle}(x) = g(x) = b_1 x + b_2 x^2 + b_3 x^3 + \dots$ if $f(g(x)) = g(f(x)) = x$.

Proposition. $f(x)$ has a compositional inverse iff $a_1 \neq 0$. In this case, the compositional inverse is unique.

Note that

$$a_1(b_1x+b_2x^2+b_3x^3+\dots)+a_2(b_1x+b_2x^2+\dots)^2+a_3(b_1x+\dots)^3+\dots = x+0x^2+0x^3+\dots$$

if and only if

$$\begin{aligned} a_1 b_1 &= 1 \\ a_1 b_2 + a_2 b_1^2 &= 0 \\ a_1 b_3 + 2a_2 b_1 b_2 + a_3 b_1^3 &= 0 \\ \dots \end{aligned}$$

Theorem (Lagrange Inversion Formula). In particular,

$$[x^n]F^{\langle -1 \rangle}(x) = \frac{1}{n}[x^{n-1}]\left(\frac{x}{F(x)}\right)^n$$

where the right-hand-side can be written equivalently as $\frac{1}{n}[x^{-1}]F(x)^{-n}$.

Exercise 1: Let $F(x) = \sum_{k \geq 1} \frac{x^k}{k!}$ and show that $F^{\langle -1 \rangle}(x) = \sum_{k \geq 1} \frac{(-1)^{k+1}}{k} x^k$.

(**Hint:** You will also recognize these as power series of familiar functions.

Exercise 2: Let $F(x) = xe^{-x}$ and we have $E_t(x) = F^{\langle -1 \rangle}(x)$. Also

$$\frac{1}{n}[x^{n-1}]\left(\frac{x}{xe^{-x}}\right)^n = \frac{1}{n}[x^{n-1}]e^{nx} = \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}.$$

Consequently, we obtain a second proof that $t_n = n^{n-1}$.