Exercises 1

We will (subjectively) indicate the difficulty level of each problem as follows:

[1] easy: most students should be able to solve it
[2] moderately difficult: many students should be able to solve it
[3] difficult: a few students should be able to solve it
[4] horrendous: no students should be able to solve it (without already knowing how)
[5] unsolved.

Further gradations are indicated by + and −. Thus a [3–] problem is about the most difficult that makes a reasonable homework exercise, and a [5–] problem is an unsolved problem that has received little attention and may not be too difficult.

NOTE. Unless explicitly stated otherwise, all graphs, posets, lattices, etc., are assumed to be *finite*.

(1) [2] Show that every region R of an arrangement $\mathcal{A}$ in $\mathbb{R}^n$ is an open convex set. Deduce that R is homeomorphic to the interior of an n-dimensional ball.

(2) [1+] Let $\mathcal{A}$ be an arrangement and $\mathrm{ess}(\mathcal{A})$ its essentialization. Show that

$$t^{\dim(\mathrm{ess}(\mathcal{A}))}\chi_{\mathcal{A}}(t) = t^{\dim(\mathcal{A})}\chi_{\mathrm{ess}(\mathcal{A})}(t).$$

(3) [2+] Let $\mathcal{A}$ be the arrangement in $\mathbb{R}^n$ with equations

$$x_1 = x_2,\ x_2 = x_3,\ \dots,\ x_{n-1} = x_n,\ x_n = x_1.$$

Compute the characteristic polynomial $\chi_{\mathcal{A}}(t)$, and compute the number $r(\mathcal{A})$ of regions of $\mathcal{A}$.

(4) [2+] Let $\mathcal{A}$ be an arrangement in $\mathbb{R}^n$ with m hyperplanes. Find the maximum possible number $f(n, m)$ of regions of $\mathcal{A}$.

(5) [2] Let $\mathcal{A}$ be an arrangement in the n-dimensional vector space V whose normals span a subspace W, and let $\mathcal{B}$ be another arrangement in V whose normals span a subspace Y. Suppose that $W \cap Y = \{0\}$. Show that

$$\chi_{\mathcal{A}\cup\mathcal{B}}(t) = t^{-n}\chi_{\mathcal{A}}(t)\chi_{\mathcal{B}}(t).$$

(6) [2] Let $\mathcal{A}$ be an arrangment in a vector space V. Suppose that $\chi_{\mathcal{A}}(t)$ is divisible by t^k but not t^{k+1}. Show that $\mathrm{rank}(\mathcal{A}) = n - k$.

(7) Let $\mathcal{A}$ be an essential arrangement in $\mathbb{R}^n$. Let Γ be the union of the bounded faces of $\mathcal{A}$.
   (a) [3] Show that Γ is contractible.
   (b) [2] Show that Γ need not be homeomorphic to a closed ball.
   (c) [2+] Show that Γ need not be starshaped. (A subset S of $\mathbb{R}^n$ is *starshaped* if there is a point $x \in S$ such that for all $y \in S$, the line segment from x to y lies in S.)
   (d) [3] Show that Γ is pure, i.e., all maximal faces of Γ have the same dimension. (This was an open problem solved by Xun Dong at the PCMI Summer Session in Geometric Combinatorics, July 11–31, 2004.)
   (e) [5] Suppose that $\mathcal{A}$ is in general position. Is Γ homeomorphic to an n-dimensional closed ball?