Exercises 3

(1) (a) [1+] Let $\chi_G(t)$ be the characteristic polynomial of the graphical arrangement $\mathcal{A}_G$. Suppose that $\chi_G(i) = 0$, where $i \in \mathbb{Z}$, $i > 1$. Show that $\chi_G(i-1) = 0$.
(b) [2] Is the same conclusion true for *any* central arrangement $\mathcal{A}$?

(2) [2] Show that if F and F' are flats of a matroid M, then so is $F \cap F'$.

(3) [2] Prove the assertion in the Note following the proof of Theorem 3.8 that an interval $[x, y]$ of a geometric lattice L is also a geometric lattice.

(4) [2+] Let $\mathcal{A}$ be an arrangement (not necessarily central). Show that there exists a geometric lattice L and an atom a of L such that $L(\mathcal{A}) \cong L - V_a$, where $V_a = \{x \in L : x \geq a\}$.

(5) [2–] Let L be a geometric lattice of rank n, and define the *truncation* $T(L)$ to be the subposet of L consisting of all elements of rank $\neq n-1$. Show that $T(L)$ is a geometric lattice.

(6) Let W_i be the number of elements of rank i in a geometric lattice (or just in the intersection poset of a central hyperplane arrangement, if you prefer) of rank n.
(a) [3] Show that for $k \leq n/2$,

$$W_1 + W_2 + \cdots + W_k \leq W_{n-k} + W_{n-k+1} + \cdots + W_{n-1}.$$

(b) [2–] Deduce from (a) and Exercise 5 that $W_1 \leq W_k$ for all $1 \leq k \leq n-1$.
(c) [5] Show that $W_i \leq W_{n-i}$ for $i < n/2$ and that the sequence $W_0, W_1, \ldots, W_n$ is unimodal. (Compare Lecture 2, Exercise 9.)

(7) [3–] Let $x \leq y$ in a geometric lattice L. Show that $\mu(x, y) = \pm 1$ if and only if the interval $[x, y]$ is isomorphic to a boolean algebra. (Use Weisner's theorem.)
Note. This problem becomes much easier using Theorem 4.12 (the Broken Circuit Theorem); see Exercise 13.

Exercises 4

(1) [2–] Let M be a matroid on a linearly ordered set. Show that $\mathrm{BC}(M) = \mathrm{BC}(\widehat{M})$, where $\widehat{M}$ is defined by equation (23).

(2) [2+] Let M be a matroid of rank at least one. Show that the coefficients of the polynomial $\chi_M(t)/(t-1)$ alternate in sign.

(3) (a) [2+] Let L be finite lattice for which every element has a unique complement. Show that L is isomorphic to a boolean algebra B_n.

(b) [3] A lattice L is *distributive* if

$$\begin{aligned} x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z) \\ x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z) \end{aligned}$$

for all $x, y, z \in L$. Let L be an infinite lattice with $\hat{0}$ and $\hat{1}$. If every element of L has a unique complement, then is L a distributive lattice?

(4) [3–] Let x be an element of a geometric lattice L. Show that the following four conditions are equivalent.

(i) x is a modular element of L.

(ii) If $x \wedge y = \hat{0}$, then

$$\mathrm{rk}(x) + \mathrm{rk}(y) = \mathrm{rk}(x \vee y).$$

(iii) If x and y are complements, then $\mathrm{rk}(x) + \mathrm{rk}(y) = n$.

(iv) All complements of x are incomparable.

(5) [2+] Let x, y be modular elements of a geometric lattice L. Show that $x \wedge y$ is also modular.

(6) [2] Let L be a geometric lattice. Prove or disprove: if x is modular in L and y is modular in the interval $[x, \hat{1}]$, then y is modular in L.

(7) [2–] Let L and L' be finite lattices. Show that if both L and L' are geometric (respectively, atomic, semimodular, modular) lattices, then so is $L \times L'$.

(8) [2] Let G be a (loopless) connected graph and $v \in V(G)$. Let $A = V(G) - v$ and $\pi = \{A, v\} \in L_G$. Suppose that whenever $av, bv \in E(G)$ we have $ab \in E(G)$. Show that π is a modular element of L_G.

(9) [2+] Generalize the previous exercise as follows. Let G be a doubly-connected graph with lattice of contractions L_G. Let $\pi \in L_G$. Show that the following two conditions are equivalent.

(a) π is a modular element of L_G.

(b) π satisfies the following two properties:

(i) At most one block B of π contains more than one vertex of G.

(ii) Let H be the subgraph induced by the block B of (i). Let K be any connected component of the subgraph induced by $G - B$, and let H_1 be the graph induced by the set of vertices in H that are connected to some vertex in K. Then H_1 is a clique (complete subgraph) of G.

(10) [2+] Let L be a geometric lattice of rank n, and fix $x \in L$. Show that

$$\chi_L(t) = \sum_{\substack{y \in L \\ x \wedge y = \hat{0}}} \mu(y) \chi_{L_y}(t) t^{n - \mathrm{rk}(x \vee y)},$$

where L_y is the image of the interval $[\hat{0}, x]$ under the map $z \mapsto z \vee y$.

(11) [2+] Let $\mathcal{I}(M)$ be the set of independent sets of a matroid M. Find another matroid N and a labeling of its points for which $\mathcal{I}(M) = \mathrm{BC}_r(N)$, the reduced broken circuit complex of N.

(12) (a) [2+] If Δ and Γ are simplicial complexes on disjoint sets A and B, respectively, then define the *join* $\Delta * \Gamma$ to be the simplicial complex on the set $A \cup B$ with faces $F \cup G$, where $F \in \Delta$ and $G \in \Gamma$. (E.g., if Γ consists of a single point then $\Delta * \Gamma$ is the *cone* over Δ. If Γ consists of two disjoint points, then $\Delta * \Gamma$ is the *suspension* of Δ.) We say that Δ and Γ are *join-factors* of $\Delta * \Gamma$. Now let M be a matroid and $S \subset M$ a modular flat, i.e., S is a modular element of L_M. Order the points of M such that if $p \in S$ and $q \notin S$, then $p < q$. Show that $\mathrm{BC}(S)$ is a join-factor of $\mathrm{BC}(M)$. Deduce that $\chi_M(t)$ is divisible by $\chi_S(t)$.

(b) [2+] Conversely, let M be a matroid and $S \subset M$. Label the points of M so that if $p \in S$ and $q \notin S$, then $p < q$. Suppose that $\mathrm{BC}(S)$ is a join-factor of $\mathrm{BC}(M)$. Show that S is modular.

(13) [2] Do Exercise 7, this time using Theorem 4.12 (the Broken Circuit Theorem).

(14) [1] Show that all geometric lattices of rank two are supersolvable.

(15) [2] Give an example of two nonisomorphic supersolvable geometric lattices of rank 3 with the same characteristic polynomials.

(16) [2] Prove Proposition 4.11: if G is a graph with blocks $G_1, \ldots, G_k$, then $L_G \cong L_{G_1} \times \cdots \times L_{G_k}$.

(17) [2+] Give an example of a nonsupersolvable geometric lattice of rank three whose characteristic polynomial has only integer zeros.

(18) [2] Let L_1 and L_2 be geometric lattices. Show that L_1 and L_2 are supersolvable if and only if $L_1 \times L_2$ is supersolvable.

(19) [3–] Let L be a supersolvable geometric lattice. Show that every interval of L is also supersolvable.

(20) [2] (a) Find the number of maximal chains of the partition lattice Π_n.

(b) Find the number of modular maximal chains of Π_n.

(21) Let M be a matroid with a linear ordering of its points. The *internal activity* of a basis B is the number of points $p \in B$ such that $p < q$ for all points $q \neq p$ not in the closure $\overline{B-p}$ of $B-p$. The *external activity* of B is the number of points $p' \in M - B$ such that $p' < q'$ for all $q' \neq p'$ contained in the unique circuit that is a subset of $B \cup \{p'\}$. Define the *Crapo beta invariant* of M by

$$\beta(M) = (-1)^{\mathrm{rk}(M)-1} \chi_M'(1),$$

where $'$ denotes differentiation.

(a) [1+] Show that $1 - \chi_M'(1) = \psi(\mathrm{BC}_r)$, the Euler characteristic of the reduced broken circuit complex of M.

(b) [3–] Show that $\beta(M)$ is equal to the number of bases of M with internal activity 0 and external activity 0.

(c) [2] Let $\mathcal{A}$ be a real central arrangement with associated matroid $M_{\mathcal{A}}$. Suppose that $\mathcal{A} = c\mathcal{A}'$ for some arrangement $\mathcal{A}'$, where $c\mathcal{A}'$ denotes the cone over $\mathcal{A}'$. Show that $\beta(M_{\mathcal{A}}) = b(\mathcal{A}')$.

(d) [2+] With $\mathcal{A}$ as in (c), let H' be a (proper) translate of some hyperplane $H \in \mathcal{A}$. Show that $\beta(M_{\mathcal{A}}) = b(\mathcal{A} \cup \{H'\})$.