LECTURE 1
Basic definitions, the intersection poset and the characteristic polynomial

1.1. Basic definitions

The following notation is used throughout for certain sets of numbers:

$\mathbb{N}$	nonnegative integers
$\mathbb{P}$	positive integers
$\mathbb{Z}$	integers
$\mathbb{Q}$	rational numbers
$\mathbb{R}$	real numbers
$\mathbb{R}_+$	positive real numbers
$\mathbb{C}$	complex numbers
$[m]$	the set $\{1, 2, \dots, m\}$ when $m \in \mathbb{N}$

We also write $[t^k]\chi(t)$ for the coefficient of t^k in the polynomial or power series $\chi(t)$. For instance, $[t^2](1+t)^4 = 6$.

A *finite hyperplane arrangement* $\mathcal{A}$ is a finite set of affine hyperplanes in some vector space $V \cong K^n$, where K is a field. We will not consider infinite hyperplane arrangements or arrangements of general subspaces or other objects (though they have many interesting properties), so we will simply use the term *arrangement* for a finite hyperplane arrangement. Most often we will take $K = \mathbb{R}$, but as we will see even if we're only interested in this case it is useful to consider other fields as well. To make sure that the definition of a hyperplane arrangement is clear, we define a *linear hyperplane* to be an $(n-1)$-dimensional subspace H of V, i.e.,

$$H = \{v \in V \,:\, \alpha \cdot v = 0\},$$

where α is a fixed nonzero vector in V and $\alpha \cdot v$ is the usual dot product:

$$(\alpha_1, \dots, \alpha_n) \cdot (v_1, \dots, v_n) = \sum \alpha_i v_i.$$

An *affine hyperplane* is a translate J of a linear hyperplane, i.e.,

$$J = \{v \in V \,:\, \alpha \cdot v = a\},$$

where α is a fixed nonzero vector in V and $a \in K$.

If the equations of the hyperplanes of $\mathcal{A}$ are given by $L_1(x) = a_1, \dots, L_m(x) = a_m$, where $x = (x_1, \dots, x_n)$ and each $L_i(x)$ is a homogeneous linear form, then we call the polynomial

$$Q_{\mathcal{A}}(x) = (L_1(x) - a_1) \cdots (L_m(x) - a_m)$$

the *defining polynomial* of $\mathcal{A}$. It is often convenient to specify an arrangement by its defining polynomial. For instance, the arrangement $\mathcal{A}$ consisting of the n coordinate hyperplanes has $Q_{\mathcal{A}}(x) = x_1 x_2 \cdots x_n$.

Let $\mathcal{A}$ be an arrangement in the vector space V. The *dimension* $\dim(\mathcal{A})$ of $\mathcal{A}$ is defined to be $\dim(V)$ $(= n)$, while the *rank* $\operatorname{rank}(\mathcal{A})$ of $\mathcal{A}$ is the dimension of the space spanned by the normals to the hyperplanes in $\mathcal{A}$. We say that $\mathcal{A}$ is *essential* if $\operatorname{rank}(\mathcal{A}) = \dim(\mathcal{A})$. Suppose that $\operatorname{rank}(\mathcal{A}) = r$, and take $V = K^n$. Let

Y be a complementary space in K^n to the subspace X spanned by the normals to hyperplanes in $\mathcal{A}$. Define

$$W = \{v \in V \,:\, v \cdot y = 0 \ \forall y \in Y\}.$$

If $\text{char}(K) = 0$ then we can simply take $W = X$. By elementary linear algebra we have

$$\text{codim}_W(H \cap W) = 1 \tag{1}$$

for all $H \in \mathcal{A}$. In other words, $H \cap W$ is a hyperplane of W, so the set $\mathcal{A}_W := \{H \cap W \,:\, H \in \mathcal{A}\}$ is an essential arrangement in W. Moreover, the arrangements $\mathcal{A}$ and $\mathcal{A}_W$ are "essentially the same," meaning in particular that they have the same intersection poset (as defined in Definition 1.1). Let us call $\mathcal{A}_W$ the *essentialization* of $\mathcal{A}$, denoted $\text{ess}(\mathcal{A})$. When $K = \mathbb{R}$ and we take $W = X$, then the arrangement $\mathcal{A}$ is obtained from $\mathcal{A}_W$ by "stretching" the hyperplane $H \cap W \in \mathcal{A}_W$ orthogonally to W. Thus if $W^\perp$ denotes the orthogonal complement to W in V, then $H' \in \mathcal{A}_W$ if and only if $H' \oplus W^\perp \in \mathcal{A}$. Note that in characteristic p this type of reasoning fails since the orthogonal complement of a subspace W can intersect W in a subspace of dimension greater than 0.

Example 1.1. Let $\mathcal{A}$ consist of the lines $x = a_1, \ldots, x = a_k$ in K^2 (with coordinates x and y). Then we can take W to be the x-axis, and $\text{ess}(\mathcal{A})$ consists of the points $x = a_1, \ldots, x = a_k$ in K.

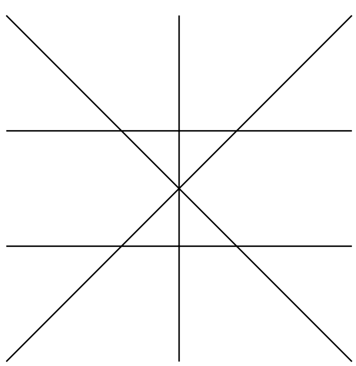
Now let $K = \mathbb{R}$. A *region* of an arrangement $\mathcal{A}$ is a connected component of the complement X of the hyperplanes:

$$X = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H.$$

Let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$, and let

$$r(\mathcal{A}) = \#\mathcal{R}(\mathcal{A}),$$

the number of regions. For instance, the arrangement $\mathcal{A}$ shown below has $r(\mathcal{A}) = 14$.



It is a simple exercise to show that every region $R \in \mathcal{R}(\mathcal{A})$ is open and convex (continuing to assume $K = \mathbb{R}$), and hence homeomorphic to the interior of an n-dimensional ball $\mathbb{B}^n$ (Exercise 1). Note that if W is the subspace of V spanned by the normals to the hyperplanes in $\mathcal{A}$, then $R \in \mathcal{R}(\mathcal{A})$ if and only if $R \cap W \in \mathcal{R}(\mathcal{A}_W)$. We say that a region $R \in \mathcal{R}(\mathcal{A})$ is *relatively bounded* if $R \cap W$ is bounded. If $\mathcal{A}$ is essential, then relatively bounded is the same as bounded. We write $b(\mathcal{A})$ for the number of relatively bounded regions of $\mathcal{A}$. For instance, in Example 1.1 take $K = \mathbb{R}$ and $a_1 < a_2 < \cdots < a_k$. Then the relatively bounded regions are the regions $a_i < x < a_{i+1}$, $1 \le i \le k-1$. In $\text{ess}(\mathcal{A})$ they become the (bounded) open intervals (a_i, a_{i+1}). There are also two regions of $\mathcal{A}$ that are not relatively bounded, viz., $x < a_1$ and $x > a_k$.

A (closed) *half-space* is a set $\{x \in \mathbb{R}^n : x \cdot \alpha \ge c\}$ for some $\alpha \in \mathbb{R}^n$, $c \in \mathbb{R}$. If H is a hyperplane in $\mathbb{R}^n$, then the complement $\mathbb{R}^n - H$ has two (open) components whose closures are half-spaces. It follows that the closure $\bar{R}$ of a region R of $\mathcal{A}$ is a finite intersection of half-spaces, i.e., a (convex) *polyhedron* (of dimension n). A bounded polyhedron is called a (convex) *polytope*. Thus if R (or $\bar{R}$) is bounded, then $\bar{R}$ is a polytope (of dimension n).

An arrangement $\mathcal{A}$ is in *general position* if

$$\begin{aligned} \{H_1, \dots, H_p\} \subseteq \mathcal{A},\ p \le n \quad &\Rightarrow \quad \dim(H_1 \cap \cdots \cap H_p) = n - p \\ \{H_1, \dots, H_p\} \subseteq \mathcal{A},\ p > n \quad &\Rightarrow \quad H_1 \cap \cdots \cap H_p = \emptyset. \end{aligned}$$

For instance, if $n = 2$ then a set of lines is in general position if no two are parallel and no three meet at a point.

Let us consider some interesting examples of arrangements that will anticipate some later material.

Example 1.2. Let $\mathcal{A}_m$ consist of m lines in general position in $\mathbb{R}^2$. We can compute $r(\mathcal{A}_m)$ using the *sweep hyperplane* method. Add a L line to $\mathcal{A}_k$ (with $\mathcal{A}_K \cup \{L\}$ in general position). When we travel along L from one end (at infinity) to the other, every time we intersect a line in $\mathcal{A}_k$ we create a new region, and we create one new region at the end. Before we add any lines we have one region (all of $\mathbb{R}^2$). Hence

$$\begin{aligned} r(\mathcal{A}_m) &= \#\text{intersections} + \#\text{lines} + 1 \\ &= \binom{m}{2} + m + 1. \end{aligned}$$

Example 1.3. The *braid arrangement* $\mathcal{B}_n$ in K^n consists of the hyperplanes

$$\mathcal{B}_n : \quad x_i - x_j = 0, \ \ 1 \le i < j \le n.$$

Thus $\mathcal{B}_n$ has $\binom{n}{2}$ hyperplanes. To count the number of regions when $K = \mathbb{R}$, note that specifying which side of the hyperplane $x_i - x_j = 0$ a point $(a_1, \dots, a_n)$ lies on is equivalent to specifying whether $a_i < a_j$ or $a_i > a_j$. Hence the number of regions is the number of ways that we can specify whether $a_i < a_j$ or $a_i > a_j$ for $1 \le i < j \le n$. Such a specification is given by imposing a linear order on the a_i's. In other words, for each permutation $w \in \mathfrak{S}_n$ (the symmetric group of all permutations of $1, 2, \dots, n$), there corresponds a region R_w of $\mathcal{B}_n$ given by

$$R_w = \{(a_1, \dots, a_n) \in \mathbb{R}^n : a_{w(1)} > a_{w(2)} > \cdots > a_{w(n)}\}.$$

Hence $r(\mathcal{B}_n) = n!$. Rarely is it so easy to compute the number of regions!

Note that the braid arrangement $\mathcal{B}_n$ is not essential; indeed, $\text{rank}(\mathcal{B}_n) = n-1$. When $\text{char}(K) \ne 2$ the space $W \subseteq K^n$ of equation (1) can be taken to be

$$W = \{(a_1, \dots, a_n) \in K^n : a_1 + \cdots + a_n = 0\}.$$

The braid arrangement has a number of "deformations" of considerable interest. We will just define some of them now and discuss them further later. All these arrangements lie in K^n, and in all of them we take $1 \le i < j \le n$. The reader who likes a challenge can try to compute their number of regions when $K = \mathbb{R}$. (Some are much easier than others.)

- *generic braid arrangement*: $x_i - x_j = a_{ij}$, where the a_{ij}'s are "generic" (e.g., linearly independent over the prime field, so K has to be "sufficiently large"). The precise definition of "generic" will be given later. (The *prime field* of K is its smallest subfield, isomorphic to either $\mathbb{Q}$ or $\mathbb{Z}/p\mathbb{Z}$ for some prime p.)
- *semigeneric braid arrangement*: $x_i - x_j = a_i$, where the a_i's are "generic."
- *Shi arrangement*: $x_i - x_j = 0, 1$ (so $n(n-1)$ hyperplanes in all).
- *Linial arrangement*: $x_i - x_j = 1$.
- *Catalan arrangement*: $x_i - x_j = -1, 0, 1$.
- *semiorder arrangement*: $x_i - x_j = -1, 1$.
- *threshold arrangement*: $x_i + x_j = 0$ (not really a deformation of the braid arrangement, but closely related).

An arrangement $\mathcal{A}$ is *central* if $\bigcap_{H\in\mathcal{A}} H \neq \emptyset$. Equivalently, $\mathcal{A}$ is a translate of a *linear arrangement* (an arrangement of linear hyperplanes, i.e., hyperplanes passing through the origin). Many other writers call an arrangement central, rather than linear, if $0 \in \bigcap_{H\in\mathcal{A}} H$. If $\mathcal{A}$ is central with $X = \bigcap_{H\in\mathcal{A}} H$, then $\operatorname{rank}(\mathcal{A}) = \operatorname{codim}(X)$. If $\mathcal{A}$ is central, then note also that $b(\mathcal{A}) = 0$ [why?].

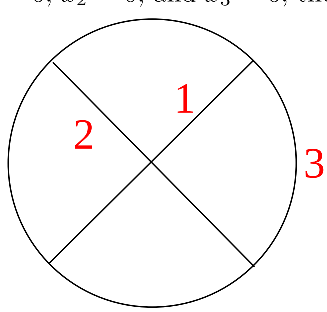
There are two useful arrangements closely related to a given arrangement $\mathcal{A}$. If $\mathcal{A}$ is a linear arrangement in K^n, then *projectivize* $\mathcal{A}$ by choosing some $H \in \mathcal{A}$ to be the hyperplane at infinity in projective space P_K^{n-1}. Thus if we regard

$$P_K^{n-1} = \{(x_1, \ldots, x_n) \,:\, x_i \in K,\ \text{not all } x_i = 0\}/\sim,$$

where $u \sim v$ if $u = \alpha v$ for some $0 \neq \alpha \in K$, then

$$H = (\{(x_1, \ldots, x_{n-1}, 0) \,:\, x_i \in K,\ \text{not all } x_i = 0\}/\sim) \cong P_K^{n-2}.$$

The remaining hyperplanes in $\mathcal{A}$ then correspond to "finite" (i.e., not at infinity) projective hyperplanes in P_K^{n-1}. This gives an arrangement $\operatorname{proj}(\mathcal{A})$ of hyperplanes in P_K^{n-1}. When $K = \mathbb{R}$, the two regions R and $-R$ of $\mathcal{A}$ become identified in $\operatorname{proj}(\mathcal{A})$. Hence $r(\operatorname{proj}(\mathcal{A})) = \frac{1}{2} r(\mathcal{A})$. When $n = 3$, we can draw $P_{\mathbb{R}}^2$ as a disk with antipodal boundary points identified. The circumference of the disk represents the hyperplane at infinity. This provides a good way to visualize three-dimensional real linear arrangements. For instance, if $\mathcal{A}$ consists of the three coordinate hyperplanes $x_1 = 0$, $x_2 = 0$, and $x_3 = 0$, then a projective drawing is given by



The line labelled i is the projectivization of the hyperplane $x_i = 0$. The hyperplane at infinity is $x_3 = 0$. There are four regions, so $r(\mathcal{A}) = 8$. To draw the incidences among all eight regions of $\mathcal{A}$, simply "reflect" the interior of the disk to the exterior:

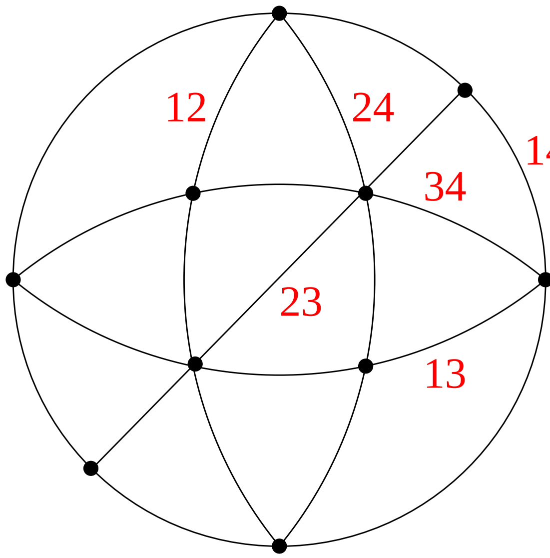
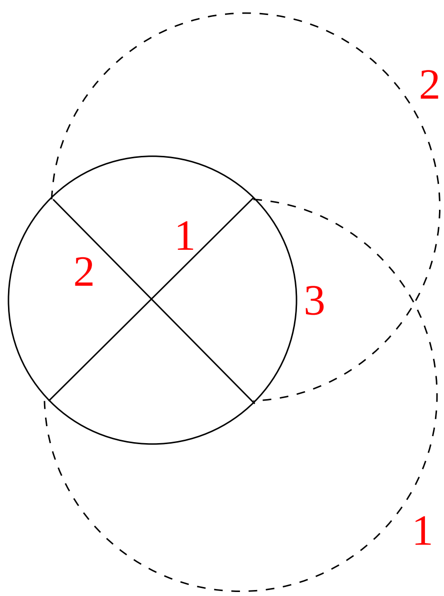


Figure 1. A projectivization of the braid arrangement $\mathcal{B}_4$



Regarding this diagram as a planar graph, the dual graph is the 3-cube (i.e., the vertices and edges of a three-dimensional cube) [why?].

For a more complicated example of projectivization, Figure 1 shows proj($\mathcal{B}_4$) (where we regard $\mathcal{B}_4$ as a three-dimensional arrangement contained in the hyperplane $x_1 + x_2 + x_3 + x_4 = 0$ of $\mathbb{R}^4$), with the hyperplane $x_i = x_j$ labelled ij, and with $x_1 = x_4$ as the hyperplane at infinity.

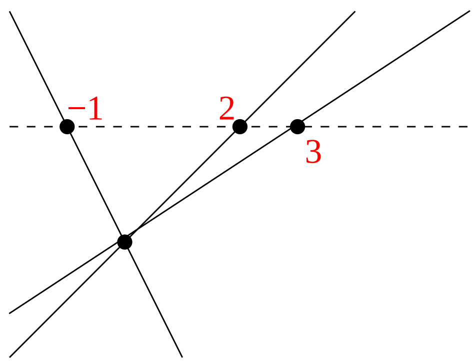
We now define an operation which is "inverse" to projectivization. Let $\mathcal{A}$ be an (affine) arrangement in K^n, given by the equations

$$L_1(x) = a_1, \quad \ldots, \quad L_m(x) = a_m.$$

Introduce a new coordinate y, and define a central arrangement $c\mathcal{A}$ (the *cone* over $\mathcal{A}$) in $K^n \times K = K^{n+1}$ by the equations

$$L_1(x) = a_1 y, \quad \ldots, \quad L_m(x) = a_m y, \quad y = 0.$$

For instance, let $\mathcal{A}$ be the arrangement in $\mathbb{R}^1$ given by $x = -1$, $x = 2$, and $x = 3$. The following figure should explain why $c\mathcal{A}$ is called a cone.



It is easy to see that when $K = \mathbb{R}$, we have $r(c\mathcal{A}) = 2r(\mathcal{A})$. In general, $c\mathcal{A}$ has the "same combinatorics as $\mathcal{A}$, times 2." See Exercise 1.

1.2. The intersection poset

Recall that a *poset* (short for partially ordered set) is a set P and a relation $\leq$ satisfying the following axioms (for all $x, y, z \in P$):

- (P1) (reflexivity) $x \leq x$
- (P2) (antisymmetry) If $x \leq y$ and $y \leq x$, then $x = y$.
- (P3) (transitivity) If $x \leq y$ and $y \leq z$, then $x \leq z$.

Obvious notation such as $x < y$ for $x \leq y$ and $x \neq y$, and $y \geq x$ for $x \leq y$ will be used throughout. If $x \leq y$ in P, then the (closed) *interval* $[x, y]$ is defined by

$$[x, y] = \{z \in P \,:\, x \leq z \leq y\}.$$

Note that the empty set $\emptyset$ is *not* a closed interval. For basic information on posets not covered here, see [**18**].

Definition 1.1. Let $\mathcal{A}$ be an arrangement in V, and let $L(\mathcal{A})$ be the set of all *nonempty* intersections of hyperplanes in $\mathcal{A}$, including V itself as the intersection over the empty set. Define $x \leq y$ in $L(\mathcal{A})$ if $x \supseteq y$ (as subsets of V). In other words, $L(\mathcal{A})$ is partially ordered by *reverse* inclusion. We call $L(\mathcal{A})$ the *intersection poset* of $\mathcal{A}$.

NOTE. The primary reason for ordering intersections by reverse inclusion rather than ordinary inclusion is Proposition 3.8. We don't want to alter the well-established definition of a geometric lattice or to refer constantly to "dual geometric lattices."

The element $V \in L(\mathcal{A})$ satisfies $x \geq V$ for all $x \in L(\mathcal{A})$. In general, if P is a poset then we denote by $\hat{0}$ an element (necessarily unique) such that $x \geq \hat{0}$ for all

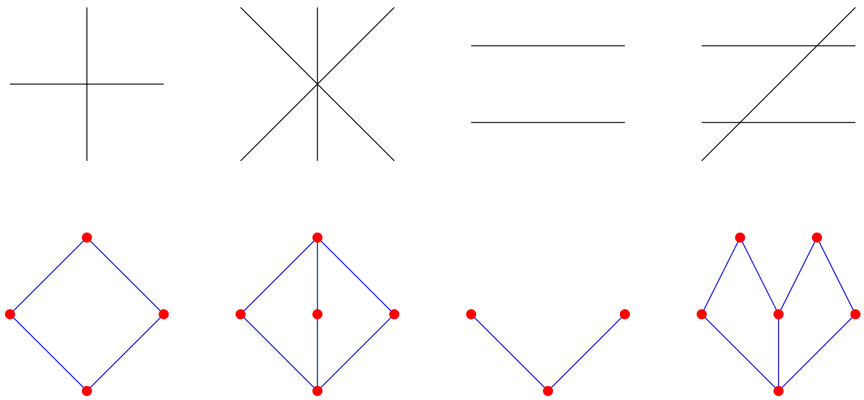


Figure 2. Examples of intersection posets

$x \in P$. We say that y *covers* x in a poset P, denoted $x \lessdot y$, if $x < y$ and no $z \in P$ satisfies $x < z < y$. Every finite poset is determined by its cover relations. The (Hasse) *diagram* of a finite poset is obtained by drawing the elements of P as dots, with x drawn lower than y if $x < y$, and with an edge between x and y if $x \lessdot y$. Figure 2 illustrates four arrangements $\mathcal{A}$ in $\mathbb{R}^2$, with (the diagram of) $L(\mathcal{A})$ drawn below $\mathcal{A}$.

A *chain of length k* in a poset P is a set $x_0 < x_1 < \cdots < x_k$ of elements of P. The chain is *saturated* if $x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k$. We say that P is *graded* of rank n if every maximal chain of P has length n. In this case P has a *rank function* $\mathrm{rk} : P \to \mathbb{N}$ defined by:

- $\mathrm{rk}(x) = 0$ if x is a minimal element of P.
- $\mathrm{rk}(y) = \mathrm{rk}(x) + 1$ if $x \lessdot y$ in P.

If $x < y$ in a graded poset P then we write $\mathrm{rk}(x, y) = \mathrm{rk}(y) - \mathrm{rk}(x)$, the *length* of the interval $[x, y]$. Note that we use the notation $\mathrm{rank}(\mathcal{A})$ for the rank of an arrangement $\mathcal{A}$ but rk for the rank function of a graded poset.

Proposition 1.1. *Let $\mathcal{A}$ be an arrangement in a vector space $V \cong K^n$. Then the intersection poset $L(\mathcal{A})$ is graded of rank equal to rank$(\mathcal{A})$. The rank function of $L(\mathcal{A})$ is given by*

$$\mathrm{rk}(x) = \mathrm{codim}(x) = n - \dim(x),$$

where dim(x) is the dimension of x as an affine subspace of V.

Proof. Since $L(\mathcal{A})$ has a unique minimal element $\hat{0} = V$, it suffices to show that (a) if $x \lessdot y$ in $L(\mathcal{A})$ then $\dim(x) - \dim(y) = 1$, and (b) all maximal elements of $L(\mathcal{A})$ have dimension $n - \mathrm{rank}(\mathcal{A})$. By linear algebra, if H is a hyperplane and x an affine subspace, then $H \cap x = x$ or $\dim(x) - \dim(H \cap x) = 1$, so (a) follows. Now suppose that x has the largest codimension of any element of $L(\mathcal{A})$, say $\mathrm{codim}(x) = d$. Thus x is an intersection of d linearly independent hyperplanes (i.e., their normals are linearly independent) $H_1, \dots, H_d$ in $\mathcal{A}$. Let $y \in L(\mathcal{A})$ with $e = \mathrm{codim}(y) < d$. Thus y is an intersection of e hyperplanes, so some H_i $(1 \le i \le d)$ is linearly independent from them. Then $y \cap H_i \neq \emptyset$ and $\mathrm{codim}(y \cap H_i) > \mathrm{codim}(y)$. Hence y is not a maximal element of $L(\mathcal{A})$, proving (b). $\square$

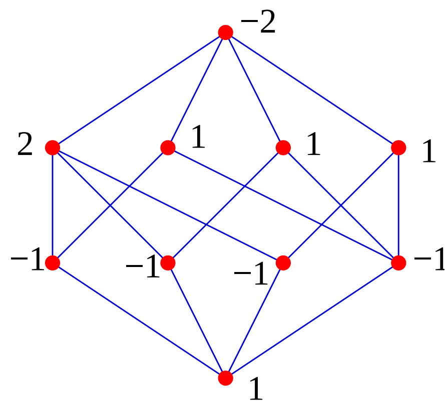


Figure 3. An intersection poset and Möbius function values

1.3. The characteristic polynomial

A poset P is *locally finite* if every interval $[x, y]$ is finite. Let $\mathrm{Int}(P)$ denote the set of all closed intervals of P. For a function $f : \mathrm{Int}(P) \to \mathbb{Z}$, write $f(x, y)$ for $f([x, y])$. We now come to a fundamental invariant of locally finite posets.

Definition 1.2. Let P be a locally finite poset. Define a function $\mu = \mu_P : \mathrm{Int}(P) \to \mathbb{Z}$, called the *Möbius function* of P, by the conditions:

$$\begin{aligned} \mu(x,x) &= 1, \text{ for all } x \in P \\ \mu(x,y) &= -\sum_{x \le z < y} \mu(x,z), \text{ for all } x < y \text{ in } P. \end{aligned} \tag{2}$$

This second condition can also be written

$$\sum_{x \le z \le y} \mu(x,z) = 0, \text{ for all } x < y \text{ in } P.$$

If P has a $\hat{0}$, then we write $\mu(x) = \mu(\hat{0}, x)$. Figure 3 shows the intersection poset L of the arrangement $\mathcal{A}$ in K^3 (for any field K) defined by $Q_{\mathcal{A}}(x) = xyz(x + y)$, together with the value $\mu(x)$ for all $x \in L$.

A important application of the Möbius function is the *Möbius inversion formula*. The best way to understand this result (though it does have a simple direct proof) requires the machinery of incidence algebras. Let $\mathcal{I}(P) = \mathcal{I}(P, K)$ denote the vector space of all functions $f : \mathrm{Int}(P) \to K$. Write $f(x, y)$ for $f([x, y])$. For $f, g \in \mathcal{I}(P)$, define the product $fg \in \mathcal{I}(P)$ by

$$fg(x,y) = \sum_{x \le z \le y} f(x,z)g(z,y).$$

It is easy to see that this product makes $\mathcal{I}(P)$ an associative $\mathbb{Q}$-algebra, with multiplicative identity δ given by

$$\delta(x,y) = \begin{cases} 1, & x = y \\ 0, & x < y. \end{cases}$$

Define the *zeta function* $\zeta \in \mathcal{I}(P)$ of P by $\zeta(x, y) = 1$ for all $x \le y$ in P. Note that the Möbius function μ is an element of $\mathcal{I}(P)$. The definition of μ (Definition 1.2) is equivalent to the relation $\mu\zeta = \delta$ in $\mathcal{I}(P)$. In any finite-dimensional algebra over a field, one-sided inverses are two-sided inverses, so $\mu = \zeta^{-1}$ in $\mathcal{I}(P)$.

Theorem 1.1. *Let P be a finite poset with Möbius function μ, and let $f, g : P \to K$. Then the following two conditions are equivalent:*

$$\begin{aligned} f(x) &= \sum_{y \geq x} g(y), \text{ for all } x \in P \\ g(x) &= \sum_{y \geq x} \mu(x,y) f(y), \text{ for all } x \in P. \end{aligned}$$

Proof. The set K^P of all functions $P \to K$ forms a vector space on which $\mathcal{I}(P)$ acts (on the left) as an algebra of linear transformations by

$$(\xi f)(x) = \sum_{y \geq x} \xi(x,y) f(y),$$

where $f \in K^P$ and $\xi \in \mathcal{I}(P)$. The Möbius inversion formula is then nothing but the statement

$$\zeta f = g \Leftrightarrow f = \mu g.$$

□

We now come to the main concept of this section.

Definition 1.3. The *characteristic polynomial* $\chi_{\mathcal{A}}(t)$ of the arrangement $\mathcal{A}$ is defined by

$$\chi_{\mathcal{A}}(t) = \sum_{x \in L(\mathcal{A})} \mu(x) t^{\dim(x)}. \tag{3}$$

For instance, if $\mathcal{A}$ is the arrangement of Figure 3, then

$$\chi_{\mathcal{A}}(t) = t^3 - 4t^2 + 5t - 2 = (t-1)^2(t-2).$$

Note that we have immediately from the definition of $\chi_{\mathcal{A}}(t)$, where $\mathcal{A}$ is in K^n, that

$$\chi_{\mathcal{A}}(t) = t^n - (\#\mathcal{A})t^{n-1} + \cdots .$$

Example 1.4. Consider the coordinate hyperplane arrangement $\mathcal{A}$ with defining polynomial $Q_{\mathcal{A}}(x) = x_1 x_2 \cdots x_n$. Every subset of the hyperplanes in $\mathcal{A}$ has a different nonempty intersection, so $L(\mathcal{A})$ is isomorphic to the *boolean algebra* B_n of all subsets of $[n] = \{1, 2, \ldots, n\}$, ordered by inclusion.

Proposition 1.2. *Let $\mathcal{A}$ be given by the above example. Then $\chi_{\mathcal{A}}(t) = (t-1)^n$.*

Proof. The computation of the Möbius function of a boolean algebra is a standard result in enumerative combinatorics with many proofs. We will give here a naive proof from first principles. Let $y \in L(\mathcal{A})$, $r(y) = k$. We claim that

$$\mu(y) = (-1)^k. \tag{4}$$

The assertion is clearly true for $\text{rk}(y) = 0$, when $y = \hat{0}$. Now let $y > \hat{0}$. We need to show that

$$\sum_{x \leq y} (-1)^{\text{rk}(x)} = 0. \tag{5}$$

The number of x such that $x \leq y$ and $\mathrm{rk}(x) = i$ is $\binom{k}{i}$, so (5) is equivalent to the well-known identity (easily proved by substituting $q = -1$ in the binomial expansion of $(q+1)^k$) $\sum_{i=0}^{k}(-1)^i\binom{k}{i} = 0$ for $k > 0$. $\square$