LECTURE 4
Broken circuits, modular elements, and supersolvability

This lecture is concerned primarily with matroids and geometric lattices. Since the intersection lattice of a central arrangement is a geometric lattice, all our results can be applied to arrangements.

4.1. Broken circuits

For any geometric lattice L and $x \leq y$ in L, we have seen (Theorem 3.10) that $(-1)^{\mathrm{rk}(x,y)}\mu(x,y)$ is a positive integer. It is thus natural to ask whether this integer has a direct combinatorial interpretation. To this end, let M be a matroid on the set $S = \{u_1, \ldots, u_m\}$. Linearly order the elements of S, say $u_1 < u_2 < \cdots < u_m$. Recall that a circuit of M is a minimal dependent subset of S.

Definition 4.10. A *broken circuit* of M (with respect to the linear ordering $\mathcal{O}$ of S) is a set $C - \{u\}$, where C is a circuit and u is the largest element of C (in the ordering $\mathcal{O}$). The *broken circuit complex* $\mathrm{BC}_{\mathcal{O}}(M)$ (or just $\mathrm{BC}(M)$ if no confusion will arise) is defined by

$$\mathrm{BC}(M) = \{T \subseteq S : T \text{ contains no broken circuit}\}.$$

Figure 1 shows two linear orderings $\mathcal{O}$ and $\mathcal{O}'$ of the points of the affine matroid M of Figure 1 (where the ordering of the points is $1 < 2 < 3 < 4 < 5$). With respect to the first ordering $\mathcal{O}$ the circuits are 123, 345, 1245, and the broken circuits are 12, 34, 124. With respect to the second ordering $\mathcal{O}'$ the circuits are 123, 145, 2345, and the broken circuits are 12, 14, 234.

It is clear that the broken circuit complex $\mathrm{BC}(M)$ is an abstract simplicial complex, i.e., if $T \in \mathrm{BC}(M)$ and $U \subseteq T$, then $U \in \mathrm{BC}(M)$. In Figure 1 we

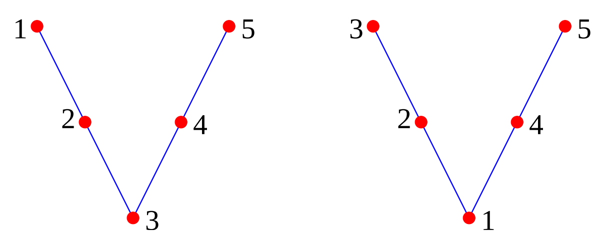
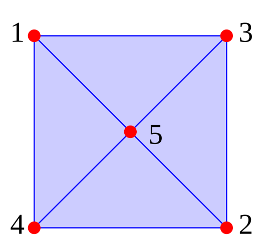
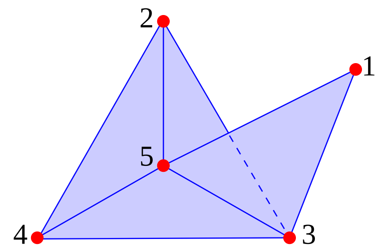


Figure 1. Two linear orderings of the matroid M of Figure 1

have $\mathrm{BC}_{\mathcal{O}}(M) = \langle 135, 145, 235, 245 \rangle$, while $\mathrm{BC}_{\mathcal{O}'}(M) = \langle 135, 235, 245, 345 \rangle$. These simplicial complexes have geometric realizations as follows:





Note that the two simplicial complexes $\mathrm{BC}_{\mathcal{O}}(M)$ and $\mathrm{BC}_{\mathcal{O}'}(M)$ are not isomorphic (as abstract simplicial complexes); in fact, their geometric realizations are not even homeomorphic. On the other hand, if $f_i(\Delta)$ denotes the number of i-dimensional faces (or faces of cardinality $i-1$) of the abstract simplicial complex Δ, then for Δ given by either $\mathrm{BC}_{\mathcal{O}}(M)$ or $\mathrm{BC}_{\mathcal{O}'}(M)$ we have

$$f_{-1}(\Delta) = 1,\ f_0(\Delta) = 5,\ f_1(\Delta) = 8,\ f_2(\Delta) = 4.$$

Note, moreover, that

$$\chi_M(t) = t^3 - 5t^2 + 8t - 4.$$

In order to generalize this observation to arbitrary matroids, we need to introduce a fair amount of machinery, much of it of interest for its own sake. First we give a fundamental formula, known as *Philip Hall's theorem*, for the Möbius function value $\mu(\hat{0}, \hat{1})$.

Lemma 4.4. *Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and with Möbius function μ. Let c_i denote the number of chains $\hat{0} = y_0 < y_1 < \cdots < y_i = \hat{1}$ in P. Then*

$$\mu(\hat{0}, \hat{1}) = -c_1 + c_2 - c_3 + \cdots.$$

Proof. We work in the incidence algebra $\mathcal{I}(P)$. We have

$$\begin{aligned} \mu(\hat{0}, \hat{1}) &= \zeta^{-1}(\hat{0}, \hat{1}) \\ &= (\delta + (\zeta - \delta))^{-1}(\hat{0}, \hat{1}) \\ &= \delta(\hat{0}, \hat{1}) - (\zeta - \delta)(\hat{0}, \hat{1}) + (\zeta - \delta)^2(\hat{0}, \hat{1}) - \cdots. \end{aligned}$$

This expansion is easily justified since $(\zeta - \delta)^k(\hat{0}, \hat{1}) = 0$ if the longest chain of P has length less than k. By definition of the product in $\mathcal{I}(P)$ we have $(\zeta - \delta)^i(\hat{0}, \hat{1}) = c_i$, and the proof follows. □

Note. Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and let $P' = P - \{\hat{0}, \hat{1}\}$. Define $\Delta(P')$ to be the set of chains of P', so $\Delta(P')$ is an abstract simplicial complex. The *reduced Euler characteristic* of a simplicial complex Δ is defined by

$$\tilde{\chi}(P) = -f_{-1} + f_0 - f_1 + \cdots,$$

where f_i is the number of i-dimensional faces $F \in \Delta$ (or $\#F = i + 1$). Comparing with Lemma 4.4 shows that

$$\mu(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P')).$$

Readers familiar with topology will know that $\tilde{\chi}(\Delta)$ has important topological significance related to the homology of Δ. It is thus natural to ask whether results

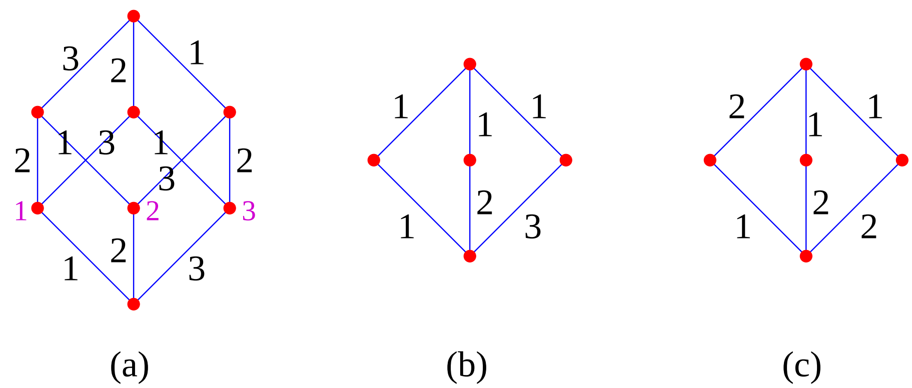


Figure 2. Three examples of edge-labelings

concerning Möbius functions can be generalized or refined topologically. Such results are part of the subject of "topological combinatorics," about which we will say a little more later.

Now let P be a finite graded poset with $\hat{0}$ and $\hat{1}$. Let

$$\mathcal{E}(P) = \{(x, y) \,:\, x \lessdot y \text{ in } P\},$$

the set of (directed) edges of the Hasse diagram of P.

Definition 4.11. An *E-labeling* of P is a map $\lambda : \mathcal{E}(P) \to \mathbb{P}$ such that if $x < y$ in P then there exists a unique saturated chain

$$C :\, x = x_0 \lessdot x_1 \lessdot x_1 \lessdot \cdots \lessdot x_k = y$$

satisfying

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k).$$

We call C the *increasing chain* from x to y.

Figure 2 shows three examples of posets P with a labeling of their edges, i.e. a map $\lambda : \mathcal{E}(P) \to \mathbb{P}$. Figure 2(a) is the boolean algebra B_3 with the labeling $\lambda(S, S \cup \{i\}) = i$. (The one-element subsets $\{i\}$ are also labelled with a small i.) For any boolean algebra B_n, this labeling is the archetypal example of an E-labeling. The unique increasing chain from S to T is obtained by adjoining to S the elements of $T - S$ one at a time in increasing order. Figures 2(b) and (c) show two different E-labelings of the same poset P. These labelings have a number of different properties, e.g., the first has a chain whose edge labels are not all different, while every maximal chain label of Figure 2(c) is a permutation of $\{1, 2\}$.

Theorem 4.11. *Let λ be an E-labeling of P, and let $x \leq y$ in P. Let μ denote the Möbius function of P. Then $(-1)^{\text{rk}(x,y)}\mu(x, y)$ is equal to the number of strictly decreasing saturated chains from x to y, i.e.,*

$$(-1)^{\text{rk}(x,y)}\mu(x, y) =$$
$$\#\{x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_k = y \,:\, \lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{k-1}, x_k)\}.$$

Proof. Since λ restricted to $[x, y]$ (i.e., to $\mathcal{E}([x, y])$) is an E-labeling, we can assume $[x, y] = [\hat{0}, \hat{1}] = P$. Let $S = \{a_1, a_2, \ldots, a_{j-1}\} \subseteq [n-1]$, with $a_1 < a_2 < \cdots < a_{j-1}$.

Define $\alpha_P(S)$ to be the number of chains $\hat{0} < y_1 < \cdots < y_{j-1} < \hat{1}$ in P such that $\mathrm{rk}(y_i) = a_i$ for $1 \leq i \leq j-1$. The function α_P is called the *flag f-vector* of P.

Claim. $\alpha_P(S)$ is the number of maximal chains $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ such that

$$\lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1}) \Rightarrow i \in S,\ 1 \leq i \leq n. \tag{27}$$

To prove the claim, let $\hat{0} = y_0 < y_1 < \cdots < y_{j-1} < y_j = \hat{1}$ with $\mathrm{rk}(y_i) = a_i$ for $1 \leq i \leq j-1$. By the definition of E-labeling, there exists a unique refinement

$$\hat{0} = y_0 = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_{a_1} = y_1 \lessdot x_{a_1+1} \lessdot \cdots \lessdot x_{a_2} = y_2 \lessdot \cdots \lessdot x_n = y_j = \hat{1}$$

satisfying

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{a_1-1}, x_{a_1})$$

$$\lambda(x_{a_1}, x_{a_1+1}) \leq \lambda(x_{a_1+1}, x_{a_1+2}) \leq \cdots \leq \lambda(x_{a_2-1}, x_{a_2})$$

$$\cdots$$

Thus if $\lambda(x_{i-1}, x_i) > \lambda(x_i, x_{i+1})$, then $i \in S$, so (27) is satisfied. Conversely, given a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ satisfying the above conditions on λ, let $y_i = x_{a_i}$. Therefore we have a bijection between the chains counted by $\alpha_P(S)$ and the maximal chains satisfying (27), so the claim follows.

Now for $S \subseteq [n-1]$ define

$$\beta_P(S) = \sum_{T \subseteq S} (-1)^{\#(S-T)} \alpha_P(T). \tag{28}$$

The function β_P is called the *flag h-vector* of P. A simple Inclusion-Exclusion argument gives

$$\alpha_P(S) = \sum_{T \subseteq S} \beta_P(T), \tag{29}$$

for all $S \subseteq [n-1]$. It follows from the claim and equation (29) that $\beta_P(T)$ is equal to the number of maximal chains $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ such that $\lambda(x_i) > \lambda(x_{i+1})$ if and only if $i \in T$. In particular, $\beta_P([n-1])$ is equal to the number of strictly decreasing maximal chains $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ of P, i.e.,

$$\lambda(x_0, x_1) > \lambda(x_1, x_2) > \cdots > \lambda(x_{n-1}, x_n).$$

Now by (28) we have

$$\begin{aligned}
\beta_P([n-1]) &= \sum_{T \subseteq [n-1]} (-1)^{n-1-\#T} \alpha_P(T) \\
&= \sum_{k \geq 1} \sum_{\hat{0} = y_0 < y_1 < \cdots < y_k = \hat{1}} (-1)^{n-k} \\
&= (-1)^n \sum_{k \geq 1} (-1)^k c_k,
\end{aligned}$$

where c_i is the number of chains $\hat{0} = y_0 < y_1 < \cdots < y_i = \hat{1}$ in P. The proof now follows from Philip Hall's theorem (Lemma 4.4). $\square$

We come to the main result of this subsection, a combinatorial interpretation of the coefficients of the characteristic polynomial $\chi_M(t)$ for any matroid M.

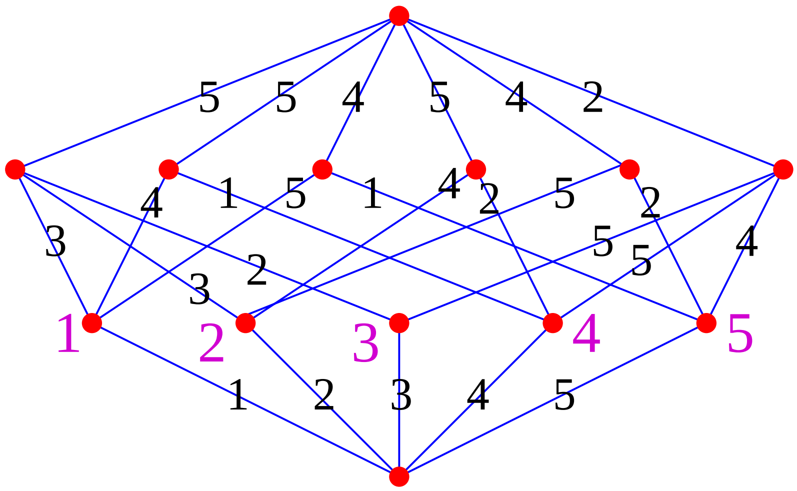


Figure 3. The edge labeling $\tilde{\lambda}$ of a geometric lattice $L(M)$

Theorem 4.12. *Let M be a matroid of rank n with a linear ordering $x_1 < x_2 < \cdots < x_m$ of its points (so the broken circuit complex $BC(M)$ is defined), and let $0 \le i \le n$. Then*

$$(-1)^i [t^{n-i}]\chi_M(t) = f_{i-1}(\mathrm{BC}(M)).$$

Proof. We may assume M is simple since the "simplification" $\widehat{M}$ has the same lattice of flats and same broken circuit complex as M (Exercise 1). The atoms x_i of $L(M)$ can then be identified with the points of M. Define a labeling $\tilde{\lambda} : \mathcal{E}(L(M)) \to \mathbb{P}$ as follows. Let $x \lessdot y$ in $L(M)$. Then set

$$\tilde{\lambda}(x, y) = \max\{i \,:\, x \vee x_i = y\}. \tag{30}$$

Note that $\tilde{\lambda}(x, y)$ is defined since $L(M)$ is atomic.

As an example, Figure 3 shows the lattice of flats of the matroid M of Figure 1 with the edge labeling (30).

Claim 1. Define $\lambda : \mathcal{E}(L(M)) \to \mathbb{P}$ by

$$\lambda(x, y) = m + 1 - \tilde{\lambda}(x, y).$$

Then λ is an E-labeling.

To prove this claim, we need to show that for all $x < y$ in $L(M)$ there is a unique saturated chain $x = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k = y$ satisfying

$$\tilde{\lambda}(y_0, y_1) \ge \tilde{\lambda}(y_1, y_2) \ge \cdots \ge \tilde{\lambda}(y_{k-1}, y_k).$$

The proof is by induction on k. There is nothing to prove for $k = 1$. Let $k > 1$ and assume the assertion for $k - 1$. Let

$$j = \max\{i \,:\, x_i \le y,\ x_i \not\le x\}.$$

For *any* saturated chain $x = z_0 \lessdot z_1 \lessdot \cdots \lessdot z_k = y$, there is some i for which $x_j \not\le z_i$ and $x_j \le z_{i+1}$. Hence $\tilde{\lambda}(z_i, z_{i+1}) = j$. Thus if $\tilde{\lambda}(z_0, z_1) \ge \cdots \ge \tilde{\lambda}(z_{k-1}, z_k)$, then $\tilde{\lambda}(z_0, z_1) = j$. Moreover, there is a unique y_1 satisfying $x = x_0 \lessdot y_1 \le y$ and $\tilde{\lambda}(x_0, y_1) = j$, viz., $y_1 = x_0 \vee x_j$. (Note that $y_1 \gtrdot x_0$ by semimodularity.)

By the induction hypothesis there exists a unique saturated chain $y_1 \lessdot y_2 \lessdot \cdots \lessdot y_k = y$ satisfying $\tilde{\lambda}(y_1, y_2) \geq \cdots \geq \tilde{\lambda}(y_{k-1}, y_k)$. Since $\tilde{\lambda}(y_0, y_1) = j > \tilde{\lambda}(y_1, y_2)$, the proof of Claim 1 follows by induction.

Claim 2. The broken circuit complex $\mathrm{BC}(M)$ consists of all chain labels $\lambda(C)$, where C is a saturated increasing chain (with respect to $\tilde{\lambda}$) from $\hat{0}$ to some $x \in L(M)$. Moreover, all such $\lambda(C)$ are distinct.

To prove the distinctness of the labels $\lambda(C)$, suppose that C is given by $\hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k$, with $\tilde{\lambda}(C) = (a_1, a_2, \ldots, a_k)$. Then $y_i = y_{i-1} \vee x_{a_i}$, so C is the only chain with its label.

Now let C and $\tilde{\lambda}(C)$ be as in the previous paragraph. We claim that the set $\{x_{a_1}, \ldots, x_{a_k}\}$ contains no broken circuit. (We don't even require that C is increasing for this part of the proof.) Write $z_i = x_{a_i}$, and suppose to the contrary that $B = \{z_{i_1}, \ldots, z_{i_j}\}$ is a broken circuit, with $1 \leq i_1 < \cdots < i_j \leq k$. Let $B \cup \{x_r\}$ be a circuit with $r > a_{i_t}$ for $1 \leq t \leq j$. Now for any circuit $\{u_1, \ldots, u_h\}$ and any $1 \leq i \leq h$ we have

$$u_1 \vee u_2 \vee \cdots \vee u_h = u_1 \vee \cdots \vee u_{i-1} \vee u_{i+1} \vee \cdots \vee u_h.$$

Thus

$$z_{i_1} \vee z_{i_2} \vee \cdots \vee z_{i_{j-1}} \vee x_r = \bigvee_{z \in B} z = z_{i_1} \vee z_{i_2} \vee \cdots \vee z_{i_j}.$$

Then $y_{i_j - 1} \vee x_r = y_{i_j}$, contradicting the maximality of the label a_{i_j}. Hence $\{x_{a_1}, \ldots, x_{a_k}\} \in \mathrm{BC}(M)$.

Conversely, suppose that $T := \{x_{a_1}, \ldots, x_{a_k}\}$ contains no broken circuit, with $a_1 < \cdots < a_k$. Let $y_i = x_{a_1} \vee \cdots \vee x_{a_i}$, and let C be the chain $\hat{0} := y_0 \lessdot y_1 \lessdot \cdots \lessdot y_k$. (Note that C is saturated by semimodularity.) We claim that $\tilde{\lambda}(C) = (a_1, \ldots, a_k)$. If not, then $y_{i-1} \vee x_j = y_i$ for some $j > a_i$. Thus

$$\mathrm{rk}(T) = \mathrm{rk}(T \cup \{x_j\}) = i.$$

Since T is independent, $T \cup \{x_j\}$ contains a circuit Q satisfying $x_j \in Q$, so T contains a broken circuit. This contradiction completes the proof of Claim 2.

To complete the proof of the theorem, note that we have shown that $f_{i-1}(\mathrm{BC}(M))$ is the number of chains $C : \hat{0} = y_0 \lessdot y_1 \lessdot \cdots \lessdot y_i$ such that $\tilde{\lambda}(C)$ is strictly increasing, or equivalently, $\lambda(C)$ is strictly decreasing. Since λ is an E-labeling, the proof follows from Theorem 4.11. □

Corollary 4.6. *The broken circuit complex* $\mathrm{BC}(M)$ *is* pure*, i.e., every maximal face has the same dimension.*

to be inserted. □

NOTE (for readers with some knowledge of topology). (a) Let M be a matroid on the linearly ordered set $u_1 < u_2 < \cdots < u_m$. Note that $F \in \mathrm{BC}(M)$ if and only if $F \cup \{u_m\} \in \mathrm{BC}(M)$. Define the *reduced broken circuit complex* $\mathrm{BC}_r(M)$ by

$$\mathrm{BC}_r(M) = \{F \in \mathrm{BC}(M) \, : \, u_m \notin F\}.$$

Thus

$$\mathrm{BC}(M) = \mathrm{BC}_r(M) * u_m,$$

the *join* of $\mathrm{BC}_r(M)$ and the vertex u_m. Equivalently, $\mathrm{BC}(M)$ is a *cone* over $\mathrm{BC}_r(M)$ with apex u_m. As a consequence, $\mathrm{BC}(M)$ is contractible and therefore has the homotopy type of a point. A more interesting problem is to determine the topological nature of $\mathrm{BC}_r(M)$. It can be shown that $\mathrm{BC}_r(M)$ has the homotopy type of a wedge of $\beta(M)$ spheres of dimension $\text{rank}(M) - 2$, where $(-1)^{\text{rank}(M)-1}\beta(M) = \chi'_M(1)$ (the derivative of $\chi_M(t)$ at $t = 1$). See Exercise 21 for more information on $\beta(M)$.

(b) [to be inserted]

As an example of the applicability of our results on matroids and geometric lattices to arrangements, we have the following purely combinatorial description of the number of regions of a real central arrangement.

Corollary 4.7. *Let $\mathcal{A}$ be a central arrangement in $\mathbb{R}^n$, and let M be the matroid defined by the normals to $H \in \mathcal{A}$, i.e., the independent sets of M are the linearly independent normals. Then with respect to any linear ordering of the points of M, $r(\mathcal{A})$ is the total number of subsets of M that don't contain a broken circuit.*

Proof. Immediate from Theorems 2.5 and 4.12. □

4.2. Modular elements

We next discuss a situation in which the characteristic polynomial $\chi_M(t)$ factors in a nice way.

Definition 4.12. An element x of a geometric lattice L is *modular* if for all $y \in L$ we have

$$\text{rk}(x) + \text{rk}(y) = \text{rk}(x \wedge y) + \text{rk}(x \vee y). \tag{31}$$

Example 4.9. Let L be a geometric lattice.

(a) $\hat{0}$ and $\hat{1}$ are clearly modular (in any finite lattice).

(b) We claim that atoms a are modular.

Proof. Suppose that $a \leq y$. Then $a \wedge y = a$ and $a \vee y = y$, so equation (31) holds. (We don't need that a is an atom for this case.) Now suppose $a \not\leq y$. By semimodularity, $\text{rk}(a \vee y) = 1 + \text{rk}(y)$, while $\text{rk}(a) = 1$ and $\text{rk}(a \wedge y) = \text{rk}(\hat{0}) = 0$, so again (31) holds. □

(c) Suppose that $\text{rk}(L) = 3$. All elements of rank 0, 1, or 3 are modular by (a) and (b). Suppose that $\text{rk}(x) = 2$. Then x is modular if and only if for all elements $y \neq x$ and $\text{rk}(y) = 2$, we have that $\text{rk}(x \wedge y) = 1$.

(d) Let $L = B_n$. If $x \in B_n$ then $\text{rk}(x) = \#x$. Moreover, for any $x, y \in B_n$ we have $x \wedge y = x \cap y$ and $x \vee y = x \cup y$. Since for any finite sets x and y we have

$$\#x + \#y = \#(x \cap y) + \#(x \cup y),$$

it follows that *every* element of B_n is modular. In other words, B_n is a *modular lattice*.

(e) Let q be a prime power and $\mathbb{F}_q$ the finite field with q elements. Define $B_n(q)$ to be the lattice of subspaces, ordered by inclusion, of the vector space $\mathbb{F}_q^n$. Note that $B_n(q)$ is also isomorphic to the intersection lattice of the arrangement of *all* linear hyperplanes in the vector space $\mathbb{F}_n(q)$. Figure 4 shows the Hasse diagrams of $B_2(3)$ and $B_3(2)$.

Note that for $x, y \in B_n(q)$ we have $x \wedge y = x \cap y$ and $x \vee y = x + y$ (subspace sum). Clearly $B_n(q)$ is atomic: every vector space is the join (sum) of its one-dimensional subspaces. Moreover, $B_n(q)$ is graded of rank n, with rank function given by $\text{rk}(x) = \dim(x)$. Since for any subspaces x and y we have

$$\dim(x) + \dim(y) = \dim(x \cap y) + \dim(x + y),$$

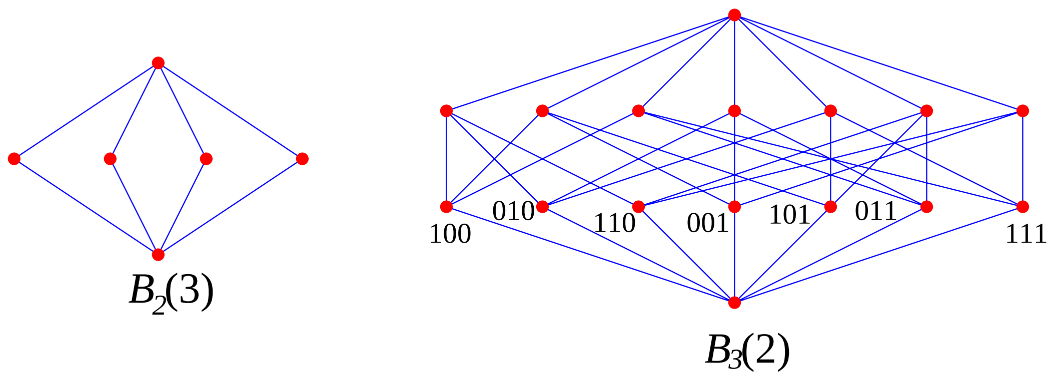


Figure 4. The lattices $B_2(3)$ and $B_3(2)$

it follows that L is a modular geometric lattice. Thus every $x \in L$ is modular.

NOTE. A *projective plane* R consists of a set (also denoted R) of points, and a collection of subsets of R, called lines, such that: (a) every two points lie on a unique line, (b) every two lines intersect in exactly one point, and (c) (non-degeneracy) there exist four points, no three of which are on a line. The *incidence lattice* $L(R)$ of R is the set of all points and lines of R, ordered by $p < L$ if $p \in L$, with $\hat{0}$ and $\hat{1}$ adjoined. It is an immediate consequence of the axioms that when R is finite, $L(R)$ is a modular geometric lattice of rank 3. It is an open (and probably intractable) problem to classify all finite projective planes. Now let P and Q be posets and define their *direct product* (or *cartesian product*) to be the set

$$P \times Q = \{(x, y) \,:\, x \in P,\ y \in Q\},$$

ordered componentwise, i.e., $(x, y) \leq (x', y')$ if $x \leq x'$ and $y \leq y'$. It is easy to see that if P and Q are geometric (respectively, atomic, semimodular, modular) lattices, then so is $P \times Q$ (Exercise 7). It is a consequence of the "fundamental theorem of projective geometry" that every finite modular geometric lattice is a direct product of boolean algebras B_n, subspace lattices $B_n(q)$ for $n \geq 3$, lattices of rank 2 with at least five elements (which may be regarded as $B_2(q)$ for any $q \geq 2$) and incidence lattices of finite projective planes.

(f) The following result characterizes the modular elements of Π_n, which is the lattice of partitions of $[n]$ or the intersection lattice of the braid arrangement $\mathcal{B}_n$.

Proposition 4.9. *A partition $\pi \in \Pi_n$ is a modular element of Π_n if and only if π has at most one nonsingleton block. Hence the number of modular elements of Π_n is $2^n - n$.*

Proof. If all blocks of π are singletons, then $\pi = \hat{0}$, which is modular by (a). Assume that π has the block A with $r > 1$ elements, and all other blocks are singletons. Hence the number $|\pi|$ of blocks of π is given by

$n-r+1$. For any $\sigma \in \Pi_n$, we have $\mathrm{rk}(\sigma) = n - |\sigma|$. Let $k = |\sigma|$ and

$$j = \#\{B \in \sigma \,:\, A \cap B \neq \emptyset\}.$$

Then $|\pi \wedge \sigma| = j + (n-r)$ and $|\pi \vee \sigma| = k - j + 1$. Hence $\mathrm{rk}(\pi) = r-1$, $\mathrm{rk}(\sigma) = n-k$, $\mathrm{rk}(\pi \wedge \sigma) = r - j$, and $\mathrm{rk}(\pi \vee \sigma) = n-k+j-1$, so π is modular.

Conversely, let $\pi = \{B_1, B_2, \ldots, B_k\}$ with $\#B_1 > 1$ and $\#B_2 > 1$. Let $a \in B_1$ and $b \in B_2$, and set

$$\sigma = \{(B_1 \cup b) - a, (B_2 \cup a) - b, B_3, \ldots, B_k\}.$$

Then

$$|\pi| = |\sigma| = k$$

$$\begin{aligned} \pi \wedge \sigma &= \{a, b, B_1 - a, B_2 - b, \ldots, B_3, \ldots, B_k\} &\Rightarrow\quad |\pi \wedge \sigma| = k+2 \\ \pi \vee \sigma &= \{B_1 \cup B_2, B_3, \ldots, B_l\} &\Rightarrow\quad |\pi \vee \sigma| = k-1. \end{aligned}$$

Hence $\mathrm{rk}(\pi) + \mathrm{rk}(\sigma) \neq \mathrm{rk}(\pi \wedge \sigma) + \mathrm{rk}(\pi \vee \sigma)$, so π is not modular. □

In a finite lattice L, a *complement* of $x \in L$ is an element $y \in L$ such that $x \wedge y = \hat{0}$ and $x \vee y = \hat{1}$. For instance, in the boolean algebra B_n every element has a unique complement. (See Exercise 3 for the converse.) The following proposition collects some useful properties of modular elements. The proof is left as an exercise (Exercises 4–5).

Proposition 4.10. *Let L be a geometric lattice of rank n.*

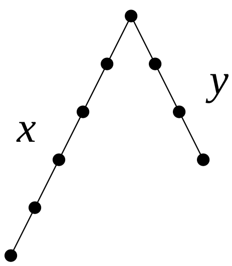
(a) *Let $x \in L$. The following four conditions are equivalent.*
  (i) *x is a modular element of L.*
  (ii) *If $x \wedge y = \hat{0}$, then $\mathrm{rk}(x) + \mathrm{rk}(y) = \mathrm{rk}(x \vee y)$.*
  (iii) *If x and y are complements, then $\mathrm{rk}(x) + \mathrm{rk}(y) = n$.*
  (iv) *All complements of x are incomparable.*
(b) *(transitivity of modularity) If x is a modular element of L and y is modular in the interval $[\hat{0}, x]$, then y is a modular element of L.*
(c) *If x and y are modular elements of L, then $x \wedge y$ is also modular.*

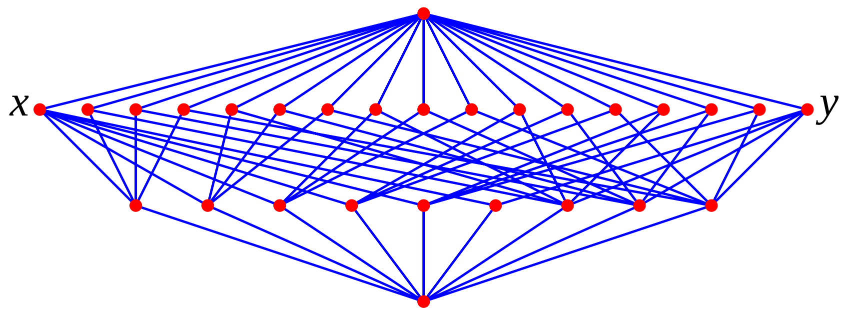
The next result, known as the *modular element factorization theorem* [**16**], is our primary reason for defining modular elements — such an element induces a factorization of the characteristic polynomial.

Theorem 4.13. *Let z be a modular element of the geometric lattice L of rank n. Write $\chi_z(t) = \chi_{[\hat{0},z]}(t)$. Then*

$$\chi_L(t) = \chi_z(t) \left[\sum_{y \,:\, y \wedge z = \hat{0}} \mu_L(y) t^{n - \mathrm{rk}(y) - \mathrm{rk}(z)} \right]. \tag{32}$$

Example 4.10. Before proceeding to the proof of Theorem 4.13, let us consider an example. The illustration below is the affine diagram of a matroid M of rank 3, together with its lattice of flats. The two lines (flats of rank 2) labelled x and y are modular by Example 4.9(c).





Hence by equation (32) $\chi_M(t)$ is divisible by $\chi_x(t)$. Moreover, any atom a of the interval $[\hat{0}, x]$ is modular, so $\chi_x(t)$ is divisible by $\chi_a(t) = t - 1$. From this it is immediate (e.g., because the characteristic polynomial $\chi_G(t)$ of any geometric lattice G of rank n begins $x^n - ax^{n-1} + \cdots$, where a is the number of atoms of G) that $\chi_x(t) = (t-1)(t-5)$ and $\chi_M(t) = (t-1)(t-3)(t-5)$. On the other hand, since y is modular, $\chi_M(t)$ is divisible by $\chi_y(t)$, and we get as before $\chi_y(t) = (t-1)(t-3)$ and $\chi_M(t) = (t-1)(t-3)(t-5)$. Geometric lattices whose characteristic polynomial factors into linear factors in a similar way due to a maximal chain of modular elements are discussed further beginning with Definition 4.13.

Our proof of Theorem 4.13 will depend on the following lemma of Greene [**11**]. We give a somewhat simpler proof than Greene.

Lemma 4.5. *Let L be a finite lattice with Möbius function μ, and let $z \in L$. The following identity is valid in the Möbius algebra $A(L)$ of L:*

$$\sigma_{\hat{0}} := \sum_{x \in L} \mu(x) x = \left(\sum_{v \leq z} \mu(v) v \right) \left(\sum_{y \wedge z = \hat{0}} \mu(y) y \right). \tag{33}$$

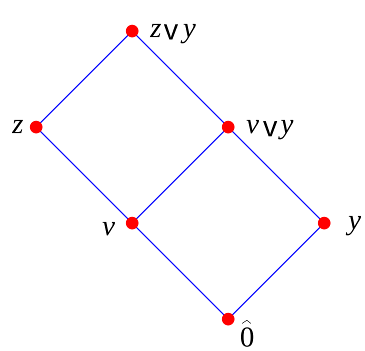
Proof. Let σ_s for $s \in L$ be given by (8). The right-hand side of equation (33) is then given by

$$\begin{aligned}
\sum_{\substack{v \leq z \\ y \wedge z = \hat{0}}} \mu(v)\mu(y)(v \vee y) &= \sum_{\substack{v \leq z \\ y \wedge z = \hat{0}}} \mu(v)\mu(y) \sum_{s \geq v \vee y} \sigma_s \\
&= \sum_s \sigma_s \sum_{\substack{v \leq s, v \leq z \\ y \leq s, y \wedge z = \hat{0}}} \mu(v)\mu(y) \\
&= \sum_s \sigma_s \left(\underbrace{\sum_{v \leq s \wedge z} \mu(v)}_{\delta_{\hat{0}, s \wedge z}} \right) \left(\sum_{\substack{y \leq s \\ y \wedge z = \hat{0}}} \mu(y) \right) \\
&= \sum_{s \wedge z = \hat{0}} \sigma_s \left(\underbrace{\sum_{\substack{y \leq s \\ y \wedge z = \hat{0} \text{ (redundant)}}} \mu(y)}_{\delta_{\hat{0}, s}} \right) \\
&= \sigma_{\hat{0}}.
\end{aligned}$$

□

Proof of Theorem 4.13. We are assuming that z is a modular element of the geometric lattice L.

Claim 1. Let $v \leq z$ and $y \wedge z = \hat{0}$ (so $v \wedge y = \hat{0}$). Then $z \wedge (v \vee y) = v$ (as illustrated below).



Proof of Claim 1. Clearly $z \wedge (v \vee y) \geq v$, so it suffices to show that $\mathrm{rk}(z \wedge (v \vee y)) \leq \mathrm{rk}(v)$. Since z is modular we have

$$\begin{aligned}
\mathrm{rk}(z \wedge (v \vee y)) &= \mathrm{rk}(z) + \mathrm{rk}(v \vee y) - \mathrm{rk}(z \vee y) \\
&= \mathrm{rk}(z) + \mathrm{rk}(v \vee y) - (\mathrm{rk}(z) + \mathrm{rk}(y) - \underbrace{\mathrm{rk}(z \wedge y)}_{0}) \\
&= \mathrm{rk}(v \vee y) - \mathrm{rk}(y) \\
&\leq (\mathrm{rk}(v) + \mathrm{rk}(y) - \underbrace{\mathrm{rk}(v \wedge y)}_{0}) - \mathrm{rk}(y) \text{ by semimodularity} \\
&= \mathrm{rk}(v),
\end{aligned}$$

proving Claim 1.

Claim 2. With v and y as above, we have $\mathrm{rk}(v \vee y) = \mathrm{rk}(v) + \mathrm{rk}(y)$.

Proof of Claim 2. By the modularity of z we have

$$\mathrm{rk}(z \wedge (v \vee y)) + \mathrm{rk}(z \vee (v \vee y)) = \mathrm{rk}(z) + \mathrm{rk}(v \vee y).$$

By Claim 1 we have $\mathrm{rk}(z \wedge (v \vee y)) = \mathrm{rk}(v)$. Moreover, again by the modularity of z we have

$$\mathrm{rk}(z \vee (v \vee y)) = \mathrm{rk}(z \vee y) = \mathrm{rk}(z) + \mathrm{rk}(y) - \mathrm{rk}(z \wedge y) = \mathrm{rk}(z) + \mathrm{rk}(y).$$

It follows that $\mathrm{rk}(v) + \mathrm{rk}(y) = \mathrm{rk}(v \vee y)$, as claimed.

Now substitute $\mu(v)v \to \mu(v)t^{\mathrm{rk}(z)-\mathrm{rk}(v)}$ and $\mu(y)y \to \mu(y)t^{n-\mathrm{rk}(y)-\mathrm{rk}(z)}$ in the right-hand side of equation (33). Then by Claim 2 we have

$$vy \to t^{n-\mathrm{rk}(v)-\mathrm{rk}(y)} = t^{n-\mathrm{rk}(v\vee y)}.$$

Now $v \vee y$ is just vy in the Möbius algebra $A(L)$. Hence if we further substitute $\mu(x)x \to \mu(x)t^{n-\mathrm{rk}(x)}$ in the left-hand side of (33), then the product will be preserved. We thus obtain

$$\underbrace{\sum_{x\in L} \mu(x)t^{n-\mathrm{rk}(x)}}_{\chi_L(t)} = \left(\underbrace{\sum_{v\leq z} \mu(v)t^{\mathrm{rk}(z)-\mathrm{rk}(v)}}_{\chi_z(t)}\right)\left(\sum_{y\wedge z=\hat{0}} \mu(y)t^{n-\mathrm{rk}(y)-\mathrm{rk}(z)}\right),$$

as desired. □

Corollary 4.8. *Let L be a geometric lattice of rank n and a an atom of L. Then*

$$\chi_L(t) = (t-1)\sum_{y\wedge a=\hat{0}} \mu(y)t^{n-1-\mathrm{rk}(y)}.$$

Proof. The atom a is modular (Example 4.9(b)), and $\chi_a(t) = t - 1$. □

Corollary 4.8 provides a nice context for understanding the operation of coning defined in Chapter 1, in particular, Exercise 2.1. Recall that if $\mathcal{A}$ is an affine arrangement in K^n given by the equations

$$L_1(x) = a_1, \ \ldots, \ L_m(x) = a_m,$$

then the cone $x\mathcal{A}$ is the arrangement in $K^n \times K$ (where y denotes the last coordinate) with equations

$$L_1(x) = a_1y, \ \ldots, L_m(x) = a_my, \ y = 0.$$

Let H_0 denote the hyperplane $y = 0$. It is easy to see by elementary linear algebra that

$$L(\mathcal{A}) \cong L(c\mathcal{A}) - \{x \in L(\mathcal{A}) : x \geq H_0\} = L(\mathcal{A}) - L(\mathcal{A}^{H_0}).$$

Now H_0 is a modular element of $L(\mathcal{A})$ (since it's an atom), so Corollary 4.8 yields

$$\chi_{c\mathcal{A}}(t) = (t-1) \sum_{y \not\geq H_0} \mu(y) t^{(n+1)-1-\mathrm{rk}(y)}$$

$$= \quad (t-1)\chi_{\mathcal{A}}(t).$$

There is a left inverse to the operation of coning. Let $\mathcal{A}$ be a nonempty linear arrangement in K^{n+1}. Let $H_0 \in \mathcal{A}$. Choose coordinates $(x_0, x_1, \ldots, x_n)$ in K^{n+1} so that $H_0 = \ker(x_0)$. Let $\mathcal{A}$ be defined by the equations

$$x_0 = 0,\ L_1(x_0, \ldots, x_n) = 0,\ \ldots,\ L_m(x_0, \ldots, x_n) = 0.$$

Define the *deconing* $c^{-1}\mathcal{A}$ (with respect to H_0) in K^n by the equations

$$L_1(1, x_1, \ldots, x_n) = 0,\ \ldots\ L_m(1, x_1, \ldots, x_n) = 0.$$

Clearly $c(c^{-1}\mathcal{A}) = \mathcal{A}$ and $L(c^{-1}\mathcal{A}) \cong L(\mathcal{A}) - \{x \in L(\mathcal{A}) : x \geq H_0\}$.

4.3. Supersolvable lattices

For some geometric lattices L, there are "enough" modular elements to give a factorization of $\chi_L(t)$ into linear factors.

Definition 4.13. A geometric lattice L is *supersolvable* if there exists a *modular maximal chain*, i.e., a maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ such that each x_i is modular. A central arrangement $\mathcal{A}$ is *supersolvable* if its intersection lattice $L_{\mathcal{A}}$ is supersolvable.

NOTE. Let $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ be a modular maximal chain of the geometric lattice L. Clearly then each x_{i-1} is a modular element of the interval $[\hat{0}, x_i]$. The converse follows from Proposition 4.10(b): if $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$ is a maximal chain for which each x_{i-1} is modular in $[\hat{0}, x_i]$, then each x_i is modular in L.

NOTE. The term "supersolvable" comes from group theory. A finite group Γ is *supersolvable* if and only if its subgroup lattice contains a maximal chain all of whose elements are normal subgroups of Γ. Normal subgroups are "nice" analogues of modular elements; see [**17**, Example 2.5] for further details.

Corollary 4.9. *Let L be a supersolvable geometric lattice of rank n, with modular maximal chain $\hat{0} = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = \hat{1}$. Let T denote the set of atoms of L, and set*

$$e_i = \#\{a \in T : a \leq x_i,\ a \not\leq x_{i-1}\}. \tag{34}$$

Then $\chi_L(t) = (t - e_1)(t - e_2) \cdots (t - e_n)$.

Proof. Since x_{n-1} is modular, we have

$$y \wedge x_{n-1} = \hat{0} \Leftrightarrow y \in T \text{ and } y \not\leq x_{n-1},\ \text{or } y = \hat{0}.$$

By Theorem 4.13 we therefore have

$$\chi_L(t) = \chi_{x_{n-1}}(t) \left[\sum_{\substack{a \in T \\ a \not\leq x_{n-1}}} \mu(a) t^{n-\mathrm{rk}(a)-\mathrm{rk}(x_{n-1})} + \mu(\hat{0}) t^{n-\mathrm{rk}(\hat{0})-\mathrm{rk}(x_{n-1})} \right].$$

Since $\mu(a) = -1$, $\mu(\hat{0}) = 1$, $\mathrm{rk}(a) = 1$, $\mathrm{rk}(\hat{0}) = 0$, and $\mathrm{rk}(x_{n-1}) = n-1$, the expression in brackets is just $t - e_n$. Now continue this with L replaced by $[\hat{0}, x_{n-1}]$ (or use induction on n). □

NOTE. The positive integers $e_1, \dots, e_n$ of Corollary 4.9 are called the *exponents* of L.

Example 4.11. (a) Let $L = B_n$, the boolean algebra of rank n. By Example 4.9(d) every element of B_n is modular. Hence B_n is supersolvable. Clearly each $e_i = 1$, so $\chi_{B_n}(t) = (t-1)^n$.

(b) Let $L = B_n(q)$, the lattice of subspaces of $\mathbb{F}_n^q$. By Example 4.9(e) every element of $B_n(q)$ is modular, so $B_n(q)$ is supersolvable. If $\left[{k \atop j}\right]$ denotes the number of j-dimensional subspaces of a k-dimensional vector space over $\mathbb{F}_q$, then

$$\begin{aligned} e_i &= \left[{i \atop 1}\right] - \left[{i-1 \atop 1}\right] \\ &= \frac{q^i - 1}{q-1} - \frac{q^{i-1} - 1}{q-1} \\ &= q^{i-1}. \end{aligned}$$

Hence

$$\chi_{B_n(q)}(t) = (t-1)(t-q)(t-q^2)\cdots(t-q^{n-1}).$$

In particular, setting $t = 0$ gives

$$\mu_{B_n(q)}(\hat{1}) = (-1)^n q^{\binom{n}{2}}.$$

NOTE. The expression $\left[{k \atop j}\right]$ is called a *q-binomial coefficient*. It is a polynomial in q with many interesting properties. For the most basic properties, see e.g. [**18**, pp. 27–30].

(c) Let $L = \Pi_n$, the lattice of partitions of the set $[n]$ (a geometric lattice of rank $n-1$). By Proposition 4.9, a maximal chain of Π_n is modular if and only if it has the form $\hat{0} = \pi_0 \lessdot \pi_1 \lessdot \cdots \lessdot \pi_{n-1} = \hat{1}$, where π_i for $i > 0$ has exactly one nonsingleton block B_i (necessarily with $i+1$ elements), with $B_1 \subset B_2 \cdots \subset B_{n-1} = [n]$. In particular, Π_n is supersolvable and has exactly $n!/2$ modular chains for $n > 1$. The atoms covered by π_i are the partitions with one nonsingleton block $\{j, k\} \subseteq B_i$. Hence π_i lies above exactly $\binom{i+1}{2}$ atoms, so

$$e_i = \binom{i+1}{2} - \binom{i}{2} = i.$$

It follows that $\chi_{\Pi_n}(t) = (t-1)(t-2)\cdots(t-n+1)$ and $\mu_{\Pi_n}(\hat{1}) = (-1)^{n-1}(n-1)!$. Compare Corollary 2.2. The polynomials $\chi_{\mathcal{B}_n}(t)$ and $\chi_{\Pi_n}(t)$ differ by a factor of t because $\mathcal{B}_n(t)$ is an arrangement in K^n of rank $n-1$. In general, if $\mathcal{A}$ is an arrangement and $\mathrm{ess}(\mathcal{A})$ its essentialization, then

$$t^{\mathrm{rk}(\mathrm{ess}(\mathcal{A}))}\chi_{\mathcal{A}}(t) = t^{\mathrm{rk}(\mathcal{A})}\chi_{\mathrm{ess}(\mathcal{A})}(t). \tag{35}$$

(See Lecture 1, Exercise 2.)

NOTE. It is natural to ask whether there is a more general class of geometric lattices L than the supersolvable ones for which $\chi_L(t)$ factors into linear factors (over $\mathbb{Z}$). There is a profound such generalization due to Terao [**22**] when L is an intersection poset of a linear arrangement $\mathcal{A}$ in K^n. Write $K[x] = K[x_1, \ldots, x_n]$ and define

$$\mathcal{T}(\mathcal{A}) = \{(p_1, \ldots, p_n) \in K[x]^n \,:\, p_i(H) \subseteq H \text{ for all } H \in \mathcal{A}\}.$$

Here we are regarding $(p_1, \ldots, p_n) : K^n \to K^n$, viz., if $(a_1, \ldots, a_n) \in K^n$, then

$$(p_1, \ldots, p_n)(a_1, \ldots, a_n) = (p_1(a_1, \ldots, a_n), \ldots, p_n(a_1, \ldots, a_n)).$$

The $K[x]$-module structure $K[x] \times \mathcal{T}(\mathcal{A}) \to \mathcal{T}(\mathcal{A})$ is given explicitly by

$$q \cdot (p_1, \ldots, p_n) = (qp_1, \ldots, qp_n).$$

Note, for instance, that we always have $(x_1, \ldots, x_n) \in \mathcal{T}(\mathcal{A})$. Since $\mathcal{A}$ is a linear arrangement, $\mathcal{T}(\mathcal{A})$ is indeed a $K[x]$-module. (We have given the most intuitive definition of the module $\mathcal{T}(\mathcal{A})$, though it isn't the most useful definition for proofs.) It is easy to see that $\mathcal{T}(\mathcal{A})$ has rank n as a $K[x]$-module, i.e., $\mathcal{T}(\mathcal{A})$ contains n, but not $n+1$, elements that are linearly independent over $K[x]$. We say that $\mathcal{A}$ is a *free* arrangement if $\mathcal{T}(\mathcal{A})$ is a free $K[x]$-module, i.e., there exist $Q_1, \ldots, Q_n \in \mathcal{T}(\mathcal{A})$ such that every element $Q \in \mathcal{T}(\mathcal{A})$ can be uniquely written in the form $Q = q_1Q_1 + \cdots + q_nQ_n$, where $q_i \in K[x]$. It is easy to see that if $\mathcal{T}(\mathcal{A})$ is free, then the basis $\{Q_1, \ldots, Q_n\}$ can be chosen to be *homogeneous*, i.e., all coordinates of each Q_i are homogeneous polynomials of the same degree d_i. We then write $d_i = \deg Q_i$. It can be shown that supersolvable arrangements are free, but there are also nonsupersolvable free arrangements. The property of freeness seems quite subtle; indeed, it is unknown whether freeness is a matroidal property, i.e., depends only on the intersection lattice $L_{\mathcal{A}}$ (regarding the ground field K as fixed). The remarkable "factorization theorem" of Terao is the following.

Theorem 4.14. *Suppose that $\mathcal{T}(\mathcal{A})$ is free with homogeneous basis $Q_1, \ldots, Q_n$. If $\deg Q_i = d_i$ then*

$$\chi_{\mathcal{A}}(t) = (t-d_1)(t-d_2)\cdots(t-d_n).$$

We will not prove Theorem 4.14 here. A good reference for this subject is [**13**, Ch. 4].

Returning to supersolvability, we can try to characterize the supersolvable property for various classes of geometric lattices. Let us consider the case of the bond lattice L_G of the graph G. A graph H with at least one edge is *doubly connected* if it is connected and remains connected upon the removal of any vertex (and all incident edges). A maximal doubly connected subgraph of a graph G is called a *block* of G. For instance, if G is a forest then its blocks are its edges. Two different blocks of G intersect in at most one vertex. Figure 5 shows a graph with eight blocks, five of which consist of a single edge. The following proposition is straightforward to prove (Exercise 16).

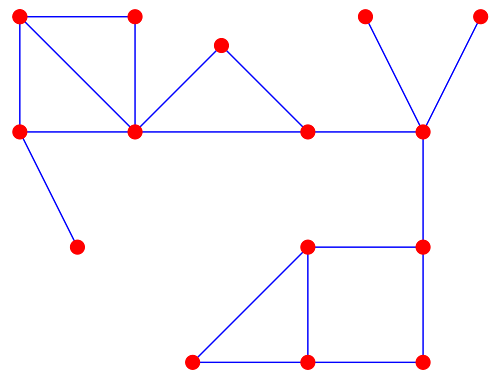


Figure 5. A graph with eight blocks

Proposition 4.11. *Let G be a graph with blocks $G_1, \dots, G_k$. Then*

$$L_G \cong L_{G_1} \times \cdots \times L_{G_k}.$$

It is also easy to see that if L_1 and L_2 are geometric lattices, then L_1 and L_2 are supersolvable if and only if $L_1 \times L_2$ is supersolvable (Exercise 18). Hence in characterizing *supersolvable graphs* G (i.e., graphs whose bond lattice L_G is supersolvable) we may assume that G is doubly connected. Note that for any connected (and hence a fortiori doubly connected) graph G, any coatom π of L_G has exactly two blocks.

Proposition 4.12. *Let G be a doubly connected graph, and let $\pi = \{A, B\}$ be a coatom of the bond lattice L_G, where $\#A \le \#B$. Then π is a modular element of L_G if and only if $\#A = 1$, say $A = \{v\}$, and the neighborhood $N(v)$ (the set of vertices adjacent to v) forms a clique (i.e., any two distinct vertices of $N(v)$ are adjacent).*

Proof. The proof parallels that of Proposition 4.9, which is a special case. Suppose that $\#A > 1$. Since G is doubly connected, there exist $u, v \in A$ and $u', v' \in B$ such that $u \ne v$, $u' \ne v'$, $uu' \in E(G)$, and $vv' \in E(G)$. Set $\sigma = \{(A \cup u') - v, (B \cup v) - u'\}$. If G has n vertices then $\mathrm{rk}(\pi) = \mathrm{rk}(\sigma) = n-2$, $\mathrm{rk}(\pi \vee \sigma) = n-1$, and $\mathrm{rk}(\pi \wedge \sigma) = n-4$. Hence π is not modular.

Assume then that $A = \{v\}$. Suppose that $av, bv \in E(G)$ but $ab \notin E(G)$. We need to show that π is not modular. Let $\sigma = \{A - \{a, b\}, \{a, b, v\}\}$. Then

$$\sigma \vee \pi = \hat{1}, \quad \sigma \wedge \pi = \{A - \{a, b\}, a, b, v\}$$

$$\mathrm{rk}(\sigma) = \mathrm{rk}(\pi) = n - 2, \quad \mathrm{rk}(\sigma \vee \pi) = n - 1, \quad \mathrm{rk}(\sigma \wedge \pi) = n - 4.$$

Hence π is not modular.

Conversely, let $\pi = \{A, v\}$. Assume that if $av, bv \in E(G)$ then $ab \in E(G)$. It is then straightforward to show (Exercise 8) that π is modular, completing the proof. □

As an immediate consequence of Propositions 4.10(b) and 4.12 we obtain a characterization of supersolvable graphs.

Corollary 4.10. *A graph G is supersolvable if and only if there exists an ordering $v_1, v_2, \dots, v_n$ of its vertices such that if $i < k$, $j < k$, $v_i v_k \in E(G)$ and $v_j v_k \in E(G)$,* *then* $v_i v_j \in E(G)$. *Equivalently, in the restriction of* G *to the vertices* $v_1, v_2, \ldots, v_i$, *the neighborhood of* v_i *is a clique.*

NOTE. Supersolvable graphs G had appeared earlier in the literature under the names *chordal*, *rigid circuit*, or *triangulated* graphs. One of their many characterizations is that any circuit of length at least four contains a chord. Equivalently, no induced subgraph of G is a k-cycle for $k \geq 4$.