

LECTURE 6

Separating Hyperplanes (preliminary version)

NOTE. This is a preliminary version of Lecture 6. Section 6.5 in particular is unfinished. (See the proof of Theorem 6.25.) Corrections and suggestions would be appreciated.

6.1. The distance enumerator

Let \mathcal{A} be a real arrangement, and let R and R' be regions of \mathcal{A} . A hyperplane $H \in \mathcal{A}$ *separates* R and R' if R and R' lie on opposite sides of H . In this chapter we will consider some results dealing with separating hyperplanes. To begin, let

$$\text{sep}(R, R') = \{H \in \mathcal{A} : H \text{ separates } R \text{ and } R'\}.$$

Define the *distance* $d(R, R')$ between the regions R and R' to be the number of hyperplanes $H \in \mathcal{A}$ that separate R and R' , i.e.,

$$d(R, R') = \#\text{sep}(R, R').$$

It is easily seen that d is a metric on the set $\mathcal{R}(\mathcal{A})$ of regions of \mathcal{A} , i.e.,

- $d(R, R') \geq 0$ for all $R, R' \in \mathcal{R}(\mathcal{A})$, with equality if and only if $R = R'$
- $d(R, R') = d(R', R)$ for all $R, R' \in \mathcal{R}(\mathcal{A})$
- $d(R, R') + d(R', R'') \geq d(R, R'')$ for all $R, R', R'' \in \mathcal{R}(\mathcal{A})$.

Now fix a region $R_0 \in \mathcal{R}(\mathcal{A})$, called the *base region*. The *distance enumerator* of \mathcal{A} (with respect to R_0) is the polynomial

$$D_{\mathcal{A}, R_0}(t) = \sum_{R \in \mathcal{R}(\mathcal{A})} t^{d(R_0, R)}.$$

We simply write $D_{\mathcal{A}}(t)$ if no confusion will result. Also define the *weak order* (with respect to R_0) of \mathcal{A} to be the partial order $W_{\mathcal{A}}$ on $\mathcal{R}(\mathcal{A})$ given by

$$R \leq R' \text{ if } \text{sep}(R_0, R) \subseteq \text{sep}(R_0, R').$$

It is easy to see that $W_{\mathcal{A}}$ is a partial ordering of $\mathcal{R}(\mathcal{A})$. The poset $W_{\mathcal{A}}$ is graded by distance from R_0 , i.e., R_0 is the $\hat{0}$ element of $\mathcal{R}(\mathcal{A})$, and all saturated chains between R_0 and R have length $d(R_0, R)$.

Figure 1 shows three arrangements in \mathbb{R}^2 , with R_0 labelled 0 and then each $R \neq R_0$ labelled $d(R_0, R)$. Under each arrangement is shown the corresponding weak order $W_{\mathcal{A}}$. The first arrangement is the braid arrangement \mathcal{B}_3 (essentialized). Here the choice of region does not affect the distance enumerator $1 + 2t + 2t^2 + t^3 = (1+t)(1+t+t^2)$ nor the weak order. On the other hand, the second two arrangements of Figure 1 are identical, but the choice of R_0 leads to different weak orders and different distance enumerators, viz., $1 + 2t + 2t^2 + t^3$ and $1 + 3t + 2t^2$.

Consider now the braid arrangement \mathcal{B}_n . We know from Example 1.3 that the regions of \mathcal{B}_n are in one-to-one correspondence with the permutations of $[n]$, viz.,

$$\begin{aligned} \mathcal{R}(\mathcal{B}_n) &\leftrightarrow \mathfrak{S}_n \\ x_{w(1)} > x_{w(2)} > \cdots > x_{w(n)} &\leftrightarrow w. \end{aligned}$$

Given $w = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$, define an *inversion* of w to be a pair (i, j) such that $i < j$ and $a_i > a_j$. Let $\ell(w)$ denote the number of inversions of w . The *inversion*

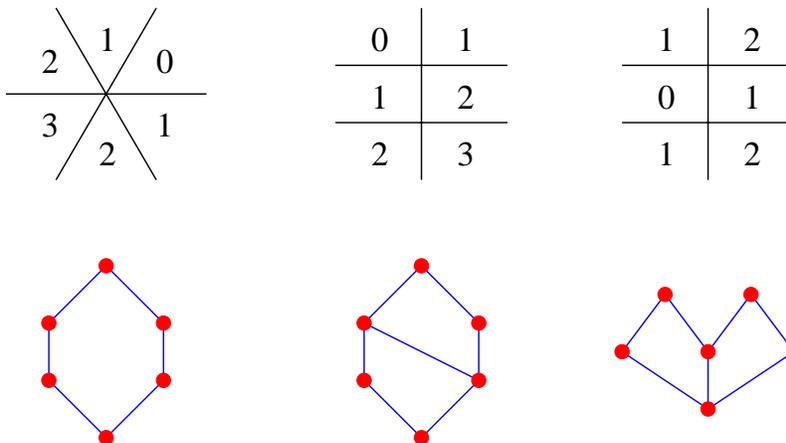


Figure 1. Examples of weak orders

sequence $\text{IS}(w)$ of w is the vector (c_1, \dots, c_n) , where

$$c_j = \#\{i : i < j, w^{-1}(j) < w^{-1}(i)\}.$$

Note that the condition $w^{-1}(j) < w^{-1}(i)$ is equivalent to i appearing to the right of j in w . For instance, $\text{IS}(461352) = (0, 0, 1, 3, 1, 4)$. The inversion sequence is a modified form of the *inversion table* or of the *code* of w , as defined in the literature, e.g., [31, p. 21][32, solution to Exer. 6.19(x)]. For our purposes the inversion sequence is the most convenient. It is clear from the definition of $\text{IS}(w)$ that if $\text{IS}(w) = (c_1, \dots, c_n)$ then $\ell(w) = c_1 + \dots + c_n$. Moreover, it is easy to see (Exercise 2) that a sequence $(c_1, \dots, c_n) \in \mathbb{N}^n$ is the inversion sequence of a permutation $w \in \mathfrak{S}_n$ if and only if $c_i \leq i - 1$ for $1 \leq i \leq n$. It follows that

$$\begin{aligned} \sum_{w \in \mathfrak{S}_n} t^{\ell(w)} &= \sum_{\substack{(c_1, \dots, c_n) \\ 0 \leq c_i \leq i-1}} t^{c_1 + \dots + c_n} \\ &= \left(\sum_{c_1=0}^0 t^{c_1} \right) \dots \left(\sum_{c_n=0}^{n-1} t^{c_n} \right) \\ (49) \quad &= 1 \cdot (1+t)(1+t+t^2) \dots (1+t+\dots+t^{n-1}), \end{aligned}$$

a standard result on permutation statistics [31, Cor. 1.3.10].

Denote by R_w the region of \mathcal{B}_n corresponding to $w \in \mathfrak{S}_n$, and choose $R_0 = R_{\text{id}}$, where $\text{id} = 12 \dots n$, the identity permutation. Suppose that $R_u, R_v \in \mathcal{R}(\mathcal{B}_n)$ such that $\text{sep}(R_0, R_v) = \{H\} \cup \text{sep}(R_0, u)$ for some $H \in \mathcal{B}_n$, $H \notin \text{sep}(R_0, R_u)$. Thus R_u and R_v are separated by a single hyperplane H , and R_0 and R_u lie on the same side of H . Suppose that H is given by $x_i = x_j$ with $i < j$. Then i and j appear consecutively in u written as a word $a_1 \dots a_n$ (since H is a bounding hyperplane of the region R_u) and i appears to the left of j (since R_0 and R_u lie on the same side of H). Thus v is obtained from u by transposing the adjacent pair ij of letters. It follows that $\ell(v) = \ell(u) + 1$. If $u(k) = i$ and we let $s_k = (k, k+1)$, the adjacent transposition interchanging k and $k+1$, then $v = us_k$.

The following result is an immediate consequence of equation (1) and mathematical induction.

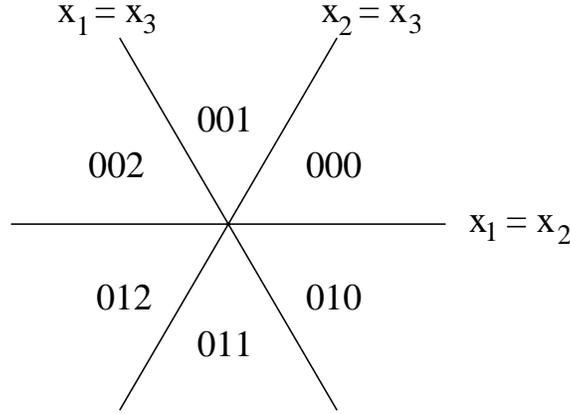


Figure 2. The inversion sequence labeling of the regions of \mathcal{B}_3

Proposition 6.18. *Let $R_0 = R_{\text{id}}$ as above. If $w \in \mathfrak{S}_n$ then $d(R_0, R_w) = \ell(w)$. Moreover,*

$$D_{\mathcal{B}_n}(t) = (1+t)(1+t+t^2)\cdots(1+t+\cdots+t^{n-1}).$$

There is a somewhat different approach to Proposition 1.1 which will be generalized to the Shi arrangement. We label each region R of \mathcal{B}_n recursively by a vector $\lambda(R) = (c_1, \dots, c_n) \in \mathbb{N}^n$ as follows.

- $\lambda(R_0) = (0, 0, \dots, 0)$
- Let e_i denote the i th unit coordinate vector in \mathbb{R}^n . If the regions R and R' of \mathcal{B}_n are separated by the single hyperplane H with the equation $x_i = x_j$, $i < j$, and if R and R_0 lie on the same side of H , then $\lambda(R') = \lambda(R) + e_j$.

Figure 2 shows the labels $\lambda(R)$ for \mathcal{B}_3 .

Proposition 6.19. *Let $w \in \mathfrak{S}_n$. Then $\lambda(R_w) = \text{IS}(w)$, the inversion sequence of w .*

Proof. The proof is a straightforward induction on $\ell(w)$. If $\ell(w) = 0$, then $w = \text{id}$ and

$$\lambda(R_{\text{id}}) = \lambda(R_0) = (0, 0, \dots, 0) = \text{IS}(\text{id}).$$

Suppose $w = a_1 \cdots a_n$ and $\ell(w) > 0$. For some $1 \leq k \leq n-1$ we must have $a_k = j > i = a_{k+1}$. Thus $\ell(ws_k) = \ell(w) - 1$. Hence by induction we may assume $\lambda(ws_k) = \text{IS}(ws_k)$. The hyperplane $x_i = x_j$ separates R_w from R_{ws_k} . Hence by the definition of λ we have

$$\lambda(R_w) = \lambda(R_{ws_k}) + e_j = \text{IS}(ws_k) + e_j.$$

By the definition of the inversion sequence we have $\text{IS}(ws_k) + e_j = \text{IS}(w)$, and the proof follows. \square

NOTE. The weak order $W_{\mathcal{B}_n}$ of the braid arrangement is an interesting poset, usually called the weak order or weak Bruhat order on \mathfrak{S}_n . For instance [14][17][30], the number of maximal chains of $W_{\mathcal{B}_n}$ is given by

$$\frac{\binom{n}{2}!}{1^{n-1}3^{n-2}5^{n-3}\cdots(2n-3)}.$$

For additional properties of $W_{\mathcal{B}_n}$, see [5].

6.2. Parking functions and tree inversions

Some beautiful enumerative combinatorics is associated with the distance enumerator of the Shi arrangement \mathcal{S}_n (for a suitable choice of R_0). The fundamental combinatorial object needed for this purpose is a parking function.

Definition 6.15. Let $n \in \mathbb{P}$. A *parking function* of length n is a sequence $(a_1, \dots, a_n) \in \mathbb{Z}^n$ whose increasing rearrangement $b_1 \leq b_2 \leq \dots \leq b_n$ satisfies $1 \leq b_i \leq i$ for $1 \leq i \leq n$. Equivalently, the sequence $(b_1 - 1, \dots, b_n - 1)$ is the inversion sequence of some permutation $w \in \mathfrak{S}_n$.

The parking function of length at most 3 are given as follows:

$$\begin{array}{ccccccc} 1 & & 11 & 12 & 21 & & \\ 111 & 112 & 121 & 211 & 113 & 131 & 311 & 122 \\ 212 & 221 & 123 & 132 & 213 & 231 & 312 & 321. \end{array}$$

The term “parking function” [21, §6] arises from the following scenario. A one-way street has parking spaces labelled $1, 2, \dots, n$ in that order. There are n cars C_1, \dots, C_n which enter the street one at a time and try to park. Each car C_i has a preferred space $a_i \in [n]$. When it is C_i 's turn to look for a space, it immediately drives to space a_i and then parks in the first available space. For instance, if $(a_1, a_2, a_3, a_4) = (2, 1, 2, 3)$, then C_1 parks in space 2, then C_2 parks in space 1, then C_3 goes to space 2 (which is occupied) parks in space 3 (the next available), and finally C_4 goes to space 3 and parks in space 4. On the other hand, if $(a_1, a_2, a_3, a_4) = (3, 1, 4, 3)$, then C_4 is unable to park, since its preferred space 3 and all subsequent spaces are already occupied. It is not hard to show (Exercise 3) that all the cars can park if and only if (a_1, \dots, a_n) is a parking function.

A basic question concerning parking functions (to be refined in Theorem 1.2) in their enumeration. The next result was first proved by Konheim and Weiss [21, §6]; we give an elegant proof due to Pollak (described in [26][16, p. 13]).

Proposition 6.20. *The number of parking functions of length n is $(n + 1)^{n-1}$.*

Proof. Arrange $n + 1$ (rather than n) parking spaces in a circle, labelled $1, \dots, n + 1$ in counterclockwise order. We still have n cars C_1, \dots, C_n with preferred spaces (a_1, \dots, a_n) , but now we can have $1 \leq a_i \leq n + 1$ (rather than $1 \leq a_i \leq n$). Each car enters the circle one at a time at their preferred space and then drives counterclockwise until encountering an empty space, in which case the car parks there. Note the following:

- All the cars can always park, since they drive in a circle and will always find an empty space.
- After all cars have parked there will be one empty space.
- The sequence (a_1, \dots, a_n) is a parking function if and only if the empty space after all the cars have parked is $n + 1$.
- If the preference sequence (a_1, \dots, a_n) produces the empty space i at the end, then the sequence $(a_1 + k, \dots, a_n + k)$ (taking entries modulo $n + 1$ so they always lie in the set $[n + 1]$) produces the empty space $i + k$ (modulo $n + 1$).

It follows that exactly one of the sequences $(a_1 + k, \dots, a_n + k)$ (modulo $n + 1$), where $1 \leq k \leq n + 1$, is a parking function. There are $(n + 1)^n$ sequences (a_1, \dots, a_n) in all, so exactly $(n + 1)^n / (n + 1) = (n + 1)^{n-1}$ are parking functions. \square

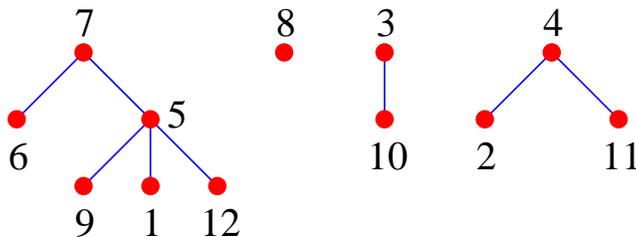


Figure 3. A rooted forest on $[12]$

Many readers will have recognized that the number $(n+1)^{n-1}$ is closely related to the enumeration of trees. Indeed, there is an intimate connection between trees and parking functions. We therefore now present some background material on trees. A *tree* on $[n]$ is a connected graph without cycles on the vertex set $[n]$. A *rooted tree* is a pair (T, i) , where T is a tree and i is a vertex of T , called the *root*. We draw trees in the standard computer science manner with the root at the top and all edges emanating downwards. A *forest* on $[n]$ is a graph F on the vertex set $[n]$ for which every (connected) component is a tree. Equivalently, F has no cycles. A *rooted forest* (also called a *planted forest*) is a forest for which every component has a root, i.e., for each tree T of the forest select a vertex i_T of T to be the root of T . A standard result in enumerative combinatorics (e.g., [32, Prop. 5.3.2]) states that the number of rooted forests on $[n]$ is $(n+1)^{n-1}$.

An *inversion* of a rooted forest F on $[n]$ is a pair (i, j) of vertices such that $i < j$ and j appears on the (unique) path from i to the root of the tree in which i occurs. Write $\text{inv}(F)$ for the number of inversions of F . For instance, the rooted forest F of Figure 3 has the inversions $(6, 7)$, $(1, 7)$, $(5, 7)$, $(1, 5)$, and $(2, 4)$, so $\text{inv}(F) = 5$.

Define the *inversion enumerator* $I_n(t)$ of rooted forests on $[n]$ by

$$I_n(t) = \sum_F t^{\text{inv}(F)},$$

where F ranges over all rooted forests on $[n]$. Figure 4 shows the 16 rooted forests on $[3]$ with their number of inversions written underneath, from which it follows that

$$I_3(t) = 6 + 6t + 3t^2 + t^3.$$

We collect below the three main results on $I_n(t)$. They are theorems in “pure” enumeration and have no direct connection with arrangements. The first result, due to Mallows and Riordan [23], gives a remarkable connection with connected graphs.

Theorem 6.21. *We have*

$$I_n(1+t) = \sum_G t^{e(G)-n},$$

where G ranges over all connected (simple) graphs on the vertex set $[0, n] = \{0, 1, \dots, n\}$ and $e(G)$ denotes the number of edges of G .

For instance,

$$I_3(1+t) = 16 + 15t + 6t^2 + t^3.$$

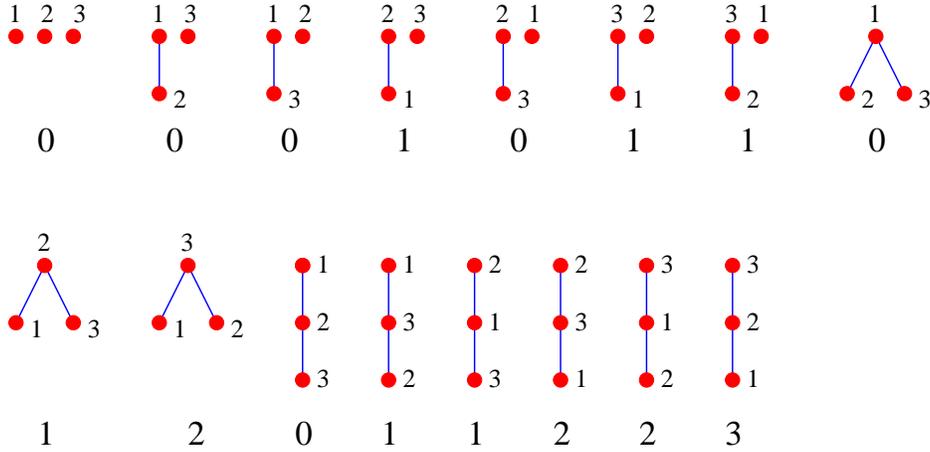


Figure 4. The 16 rooted forests on [3] and their number of inversions

Thus, for instance, there are 15 connected graphs on $[0, 3]$ with four edges. Three of these are 4-cycles and twelve consist of a triangle with an incident edge. The enumeration of connected graphs is well-understood [32, Exam. 5.2.1]. In particular, if

$$C_n(t) = \sum_G t^{e(G)},$$

where G ranges over all connected (simple) graphs on $[n]$, then

$$(50) \quad \sum_{n \geq 0} C_n(t) \frac{x^n}{n!} = \log \sum_{n \geq 1} (1+t)^{\binom{n}{2}} \frac{x^n}{n!}.$$

Thus Theorem 1.1 “determines” $I_n(t)$. There is an alternative way to state this result that doesn’t involve the logarithm function.

Corollary 6.13. *We have*

$$(51) \quad \sum_{n \geq 0} I_n(t) (t-1)^n \frac{x^n}{n!} = \frac{\sum_{n \geq 0} t^{\binom{n+1}{2}} \frac{x^n}{n!}}{\sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!}}$$

The third result, due to Kreweras [22], connects inversion enumerators with parking functions. Let PF_n denote the set of parking function of length n .

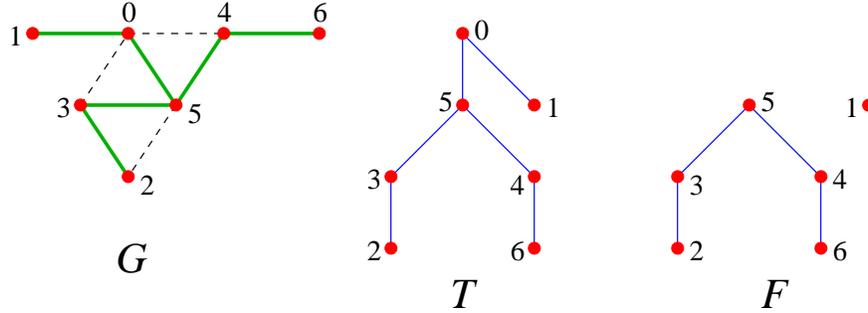
Theorem 6.22. *Let $n \geq 1$. Then*

$$(52) \quad t^{\binom{n}{2}} I_n(1/t) = \sum_{(a_1, \dots, a_n) \in \text{PF}_n} t^{a_1 + \dots + a_n - n}.$$

We now give proofs of Theorem 1.1, Corollary 1.1, and Theorem 1.2.

Proof of Theorem 1.1 (sketch). The following elegant proof is due to Gessel and Wang [18]. Let G be a connected graph on $[0, n]$. Start at vertex 0 and let T be the “depth-first spanning tree,” i.e., move to the largest unvisited neighbor or else (if there is no unvisited neighbor) backtrack. The edges traversed when all

vertices are visited are the edges of the spanning tree T . Remove the vertex 0 and root the trees that remain at the neighbors of 0. Denote this rooted forest by F_G .



Given a spanning forest F on $[n]$, what connected graphs G on $[0, n]$ satisfy $F = F_G$? The answer, whose straightforward verification we leave to the reader, is the following. Add the vertex 0 to F and connect it the roots of F , obtaining T . Clearly G consists of T with some added edges ij . The edge ij can be added to T if and only if the path from 0 to j contains i (or vice versa), and if i' is the next vertex after i on the path from i to j , then (j, i') is an inversion of F . Thus each inversion of F corresponds to a possible edge that can be added to T , and these edges can be added or not added independently. It follows that

$$\begin{aligned} \sum_{G: F=F_G} t^{e(G)} &= t^{e(T)}(1+t)^{\text{inv}(F)} \\ &= t^n(1+t)^{\text{inv}(F)}. \end{aligned}$$

Summing on all rooted forests F on $[n]$ gives

$$\begin{aligned} \sum_G t^{e(G)} &= t^n \sum_F (1+t)^{\text{inv}(F)} \\ &= t^n I_n(1+t), \end{aligned}$$

where G ranges over all connected graphs on $[0, n]$. □

Proof of Corollary 1.1. By equation (2) and Theorem 1.1 we have

$$\sum_{n \geq 0} t^{n-1} I_{n-1}(1+t) \frac{x^n}{n!} = \log \sum_{n \geq 0} (1+t)^{\binom{n}{2}} \frac{x^n}{n!}.$$

Substituting $t - 1$ for t gives

$$\sum_{n \geq 0} (t-1)^{n-1} I_{n-1}(t) \frac{x^n}{n!} = \log \sum_{n \geq 0} t^{\binom{n}{2}} \frac{x^n}{n!}.$$

Now differentiate both sides with respect to x to obtain equation (3). □

Proof of Theorem 1.2. Let

$$\begin{aligned} J_n(t) &= \sum_{(a_1, \dots, a_n) \in \text{PF}_n} t^{\binom{n}{2} - \sum (a_i - 1)} \\ &= \sum_{(a_1, \dots, a_n) \in \text{PF}_n} t^{\binom{n+1}{2} - \sum a_i}. \end{aligned}$$

Claim #1:

$$(53) \quad J_{n+1}(t) = \sum_{i=0}^n \binom{n}{i} (1 + t + t^2 + \cdots + t^i) J_i(t) J_{n-i}(t).$$

Proof of claim. Choose $0 \leq i \leq n$, and let S be an i -element subset of $[n]$. Choose also $\alpha \in \text{PF}_i$, $\beta \in \text{PF}_{n-i}$, and $0 \leq j \leq i$. Form a vector $\gamma = (\gamma_1, \dots, \gamma_{n+1})$ by placing α at the positions indexed by S , placing $(\beta_1 + i + 1, \dots, \beta_{n-i} + i + 1)$ at the positions indexed by $[n] - S$, and placing $j + 1$ at position $n + 1$. For instance, suppose $n = 7$, $i = 3$, $S = \{2, 3, 6\}$, $\alpha = (1, 2, 1)$, $\beta = (2, 1, 4, 2)$, and $j = 1$. Then $\gamma = (6, 1, 2, 5, 8, 1, 6, 2) \in \text{PF}_8$. It is easy to check that in general $\gamma \in \text{PF}_{n+1}$. Note that

$$\sum_{k=1}^{n+1} \gamma_k = \sum_{k=1}^i \alpha_k + \sum_{k=1}^{n-i} \beta_k + (n - i)(i + 1) + j + 1,$$

so

$$\binom{n+2}{2} - \sum \gamma_k = \binom{i+1}{2} - \sum \alpha_k + \binom{n-i+1}{2} - \sum \beta_k + i - j.$$

Equation (5) then follows if the map $(i, S, \alpha, \beta, j) \mapsto \gamma$ is a bijection, i.e., given $\gamma \in \text{PF}_{n+1}$, we can uniquely obtain (i, S, α, β, j) so that $(i, S, \alpha, \beta, j) \mapsto \gamma$. Now given γ , note that $i + 1$ is the largest number that can replace γ_{n+1} so that we still have a parking function. Once i is determined, the rest of the argument is clear, proving the claim.

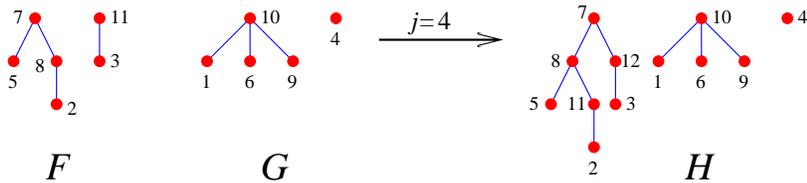
NOTE. Several bijections are known between the set of all rooted forests F on $[n]$ (or rooted trees on $[0, n]$) and the set PF_n of all parking functions (a_1, \dots, a_n) of length n , but none of them have the property that $\text{inv}(F) = a_1 + \cdots + a_n - n$. Hence a direct bijective proof of Theorem 1.2 is not known. It would be interesting to find such a proof (Exercise 4).

Claim #2:

$$(54) \quad I_{n+1}(t) = \sum_{i=0}^n \binom{n}{i} (1 + t + t^2 + \cdots + t^i) I_i(t) I_{n-i}(t).$$

Proof of claim. We give a proof due to G. Kreweras [22]. Let F be a rooted forest on $S \subseteq [n]$, $\#S = i$, and let G be a rooted forest on $\bar{S} = [n] - S$. Let $u_1 < \cdots < u_i$ be the vertices of F , and set $u_{i+1} = n + 1$. Choose $1 \leq j \leq i + 1$. For all $m \geq j$ replace u_m by u_{m+1} . (If $j = i + 1$, then do nothing.) This gives a labelled forest F' on $(S \cup \{n + 1\}) - \{u_j\}$. Let T' be the labelled tree obtained from F' by adjoining the root u_j and connecting it to the roots of F' . Keep G the same. We obtain a rooted forest H on $[n + 1]$ satisfying

$$\text{inv}(H) = j - 1 + \text{inv}(F) + \text{inv}(G).$$



This process gives a bijection $(S, F, G, j) \mapsto H$, where $S \subseteq [n]$, F is a rooted forest on S , G is a rooted forest on \bar{S} , $1 \leq j \leq 1 + \#S$, and H is a rooted forest on $[n + 1]$. Hence

$$\sum_{i=0}^n \sum_{\substack{S \subseteq [n] \\ \#S=i}} I_i(t) I_{n-i}(t) (1 + t + \cdots + t^i) = I_{n+1}(t),$$

and the claim follows.

The initial conditions $I_0(t) = J_0(t) = 1$ agree, so by the two claims we have $I_n(t) = J_n(t)$ for all $n \geq 0$. The proof of equation (4) follows by substituting $1/t$ for t . \square

6.3. The distance enumerator of the Shi arrangement

Recall that the Shi arrangement \mathcal{S}_n is given by the defining polynomial

$$Q_{\mathcal{S}_n} = \prod_{1 \leq i < j \leq n} (x_i - x_j)(x_i - x_j - 1).$$

Let $K = \mathbb{R}$, and let R_0 denote the region

$$(55) \quad x_1 > x_2 > \cdots > x_n > x_1 - 1,$$

so $x \in R_0$ if and only if $0 \leq x_i - x_j \leq 1$ for all $i < j$. We define a labeling $\lambda : \mathcal{R}(\mathcal{S}_n) \rightarrow \mathbb{N}^n$ of the regions of \mathcal{S}_n as follows.

- $\lambda(R_0) = (0, 0, \dots, 0)$
- If the regions R and R' of \mathcal{S}_n are separated by the single hyperplane H with the equation $x_i = x_j$, $i < j$, and if R and R_0 lie on the same side of H , then $\lambda(R') = \lambda(R) + e_j$ (exactly as for the braid arrangement).
- If the regions R and R' of \mathcal{B}_n are separated by the single hyperplane H with the equation $x_i = x_j + 1$, $i < j$, and if R and R_0 lie on the same side of H , then $\lambda(R') = \lambda(R) + e_i$.

Note that the labeling λ is well-defined, since $\lambda(R)$ depends only on $\text{sep}(R_0, R)$. Figure 5 shows the labeling λ for the case $n = 3$.

Theorem 6.23. *All labels $\lambda(R)$, $R \in \mathcal{R}(\mathcal{S}_n)$, are distinct, and*

$$\text{PF}_n = \{(a_1 + 1, \dots, a_n + 1) : (a_1, \dots, a_n) = \lambda(R) \text{ for some } R \in \mathcal{R}(\mathcal{S}_n)\}.$$

In other words, the labels $\lambda(R)$ for $R \in \mathcal{R}(\mathcal{S}_n)$ are obtained from the labels $\lambda(R)$ for $R \in \mathcal{R}(\mathcal{B}_n)$ by permuting coordinates in all possible ways. This remarkable fact seems much more difficult to prove than the corresponding result for \mathcal{B}_n , viz., the labels $\lambda(R)$ for \mathcal{B}_n consist of the sequences (a_1, \dots, a_n) with $0 \leq a_i \leq i - 1$ (an immediate consequence of Proposition 1.2 and Exercise 2).

Proof of Theorem 1.3 (sketch). An *antichain* \mathcal{I} of proper intervals of $[n]$ is a collection of intervals $[i, j] = \{i, i + 1, \dots, j\}$ with $1 \leq i < j \leq n$ such that if $I, I' \in \mathcal{I}$ and $I \subseteq I'$, then $I = I'$. For instance, there are five antichains of proper intervals of $[3]$, namely (writing ij for $[i, j]$)

$$\emptyset, \quad \{12\}, \quad \{23\}, \quad \{12, 23\}, \quad \{13\}.$$

In general, the number of antichains of proper intervals of $[n]$ is the Catalan number C_n (immediate from [32, Exer. 6.19(bbb)], though this fact is not relevant here).

This corresponds to the region

$$x_5 > x_2 > x_1 > x_7 > x_6 > x_9 > x_3 > x_4 > x_8$$

$$x_5 - x_7 < 1, x_2 - x_3 < 1, x_7 - x_8 < 1.$$

This region is separated from R_0 by the hyperplanes

$$x_5 = x_2, x_5 = x_1, \dots \quad (13 \text{ in all})$$

$$x_5 = x_6 + 1, x_5 = x_9 + 1, \dots \quad (7 \text{ in all}).$$

Let $\lambda(w, \mathfrak{J}, w(i))$ be the number of integers j such that (i, j) is either of type 1 or type 2. Thus

$$\lambda(R) = (\lambda(w, \mathfrak{J}, 1), \dots, \lambda(w, \mathfrak{J}, n)).$$

For the example above we have $\lambda(R) = (2, 3, 0, 0, 7, 2, 3, 0, 3)$. For instance, the entry $\lambda(w, \mathfrak{J}, 5) = 7$ corresponds to the seven pairs 12, 13, 17, 18 (type 1) and 15, 16, 19 (type 2).

Clearly $\lambda(R) + (1, 1, \dots, 1) \in \text{PF}_n$, since $\lambda(w, \mathfrak{J}, w(i)) \leq n - i$ (the number of elements to the right of $w(i)$ in w).

Key lemma. Let X be an r -element subset of $[n]$, and let $v = v_1 \cdots v_r$ be a permutation of X . Let \mathfrak{J} be an antichain of proper intervals $[a, b]$, where $v_a < v_b$. Suppose that the pair (i, j) is either of type 1 or type 2. Then

$$\lambda(v, \mathfrak{J}, v_i) > \lambda(v, \mathfrak{J}, v_j).$$

The proof of this lemma is straightforward and is left to the reader. For the example above, writing $\lambda(R) = (\lambda_1, \dots, \lambda_9) = (5, 2, 1, 7, 6, 9, 3, 4, 8)$, the above lemma implies that

- (a) $\lambda_5 > \lambda_2, \lambda_5 > \lambda_1, \lambda_5 > \lambda_3, \lambda_5 > \lambda_4, \lambda_2 > \lambda_1, \lambda_7 > \lambda_6, \lambda_7 > \lambda_3, \lambda_7 > \lambda_4,$
 $\lambda_6 > \lambda_3, \lambda_6 > \lambda_4, \lambda_9 > \lambda_3, \lambda_9 > \lambda_4, \lambda_9 > \lambda_8$
- (b) $\lambda_5 > \lambda_6, \lambda_5 > \lambda_9, \lambda_5 > \lambda_8, \lambda_2 > \lambda_4, \lambda_2 > \lambda_8, \lambda_1 > \lambda_4, \lambda_1 > \lambda_8.$

The crux of the proof of Theorem 1.3 is to show that given $\alpha + (1, 1, \dots, 1) \in \text{PF}_n$, there is a unique region $R \in \mathcal{R}(\mathcal{S}_n)$ satisfying $\lambda(R) = \alpha$. We will illustrate the construction of R from α with the example $\alpha = (2, 3, 0, 0, 7, 2, 3, 0, 3)$. We build up the pair (w, \mathfrak{J}) representing R one step at a time. First let v be the permutation of $[n]$ obtained from “standardizing” α from right-to-left. This means replacing the 0’s in α with $1, 2, \dots, m_1$ from right-to-left, then replacing the 1’s in α with $m_1 + 1, m_1 + 2, \dots, m_2$ from right-to-left, etc. Let $v^{-1} = (t_1, \dots, t_n)$. For our example, we have

$$\begin{array}{rcccccccc} \alpha & = & 2 & 3 & 0 & 0 & 7 & 2 & 3 & 0 & 3 \\ v & = & 5 & 8 & 3 & 2 & 9 & 4 & 7 & 1 & 6 \\ v^{-1} & = & 8 & 4 & 3 & 6 & 1 & 9 & 7 & 2 & 5 \end{array}$$

Next we insert t_1, \dots, t_n from left-to-right into w . From α we can read off where t_i is inserted. After inserting t_i , we also record which of the positions of the elements so far inserted belong to some interval $I \in \mathfrak{J}$. We can also determine from α the unique way to do this. The best way to understand this insertion technique is to practice with some examples. Figure 6 illustrates the steps in the insertion process for our current example. These steps are explained as follows.

- (1) First insert 8.

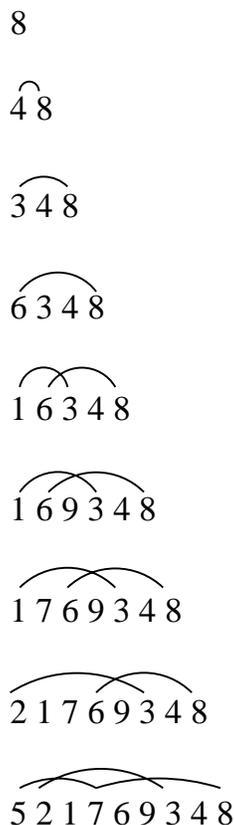


Figure 6. Constructing a valid pair (w, \mathcal{J}) from the parking function $\alpha = (2, 3, 0, 0, 7, 2, 3, 0, 3)$

- (2) Insert 4. Since $\alpha_8 = 0$, 4 appears to the left of 8, so have the partial permutation 48. We now must decide whether the positions of 4 and 8 belong to some interval $I \in \mathcal{J}$. (In other words, in the pictorial representation of (w, \mathcal{J}) , will 4 and 8 lie under some arc?) By the first term on the right-hand side of (8), we would have $\alpha_4 \geq 1$ if there were no such I . Since $\alpha_4 = 0$, we obtain the second row of Figure 6.
- (3) Insert 3. As in the previous step, we obtain 348 with a single arc over all three terms.
- (4) Insert 6. Suppose we inserted it after the 3, obtaining 3648, with a single arc over all four terms (since 3 and 8 have already been determined to lie under a single arc). We have $\alpha_6 = 2$, but the contribution so far (of 3648 with an arc over all four terms) to α_6 is 1. Thus later we must insert some j to the right of 6 so that the pair $(6, j)$ is of type 1 or type 2. By the lemma, we would have $\lambda(w, \mathcal{J}, 6) > \lambda(w, \mathcal{J}, j)$, contradicting that we are inserting elements in order of increasing α_i 's. Similarly 3468 and 3486 are excluded, so 6 must be inserted at the left, yielding 6348. If the arc over 4,6,8 is not extended to 6, then we would have $\alpha_6 \geq 3$. Hence we obtain the fourth row of Figure 6.

- (5) Insert 1. Using the lemma we obtain 16348. Since $\alpha_1 = 2$, there is an arc over 1 and two other elements to the right to 1. This gives the fifth row of Figure 6.
- (6) Insert 9. Placing 9 before 1 or 6 yields $\alpha_9 \geq 4$, contradicting $\alpha_9 = 3$. Placing 9 after 3,4, or 8 is excluded by the lemma. Hence we get the sixth row of Figure 6.
- (7) Insert 7. Placing 7 at the beginning yields four terms $j < 7$ appearing to the right of 7, giving $\alpha_7 \geq 4$, a contradiction. Placing 7 after 6,9,3,4,8 will violate the lemma, so we get the partial permutation 1769348. In order that $\alpha_7 = 3$, we must have 7 and 8 appearing under the same arc. Hence the arc from 6 to 8 must be extended to 7, yielding row seven of Figure 6.
- (8) Insert 2 and 5. By now we hope it is clear that there is always a unique way to proceed.

The uniqueness of the above procedure shows that the map from the regions R of \mathcal{S}_n (or the valid pairs (w, \mathcal{J}) that index the regions) to parking functions α is injective. Since the number of valid pairs and number of parking functions are both $(n+1)^{n-1}$, the map is bijective, completing the (sketched) proof. In fact, it's not hard to show surjectivity directly, i.e., that the above procedure produces a valid pair (w, \mathcal{J}) for *any* parking function, circumventing the need to know that $r(\mathcal{S}_n) = \#\text{PF}_n$ in advance. \square

Corollary 6.14. *The distance enumerator of \mathcal{S}_n is given by*

$$(57) \quad D_{\mathcal{S}_n}(t) = \sum_{(a_1, \dots, a_n) \in \text{PF}_n} t^{a_1 + \dots + a_n - n}.$$

Proof. It is immediate from the definition of the labeling $\lambda : \mathcal{R}(\mathcal{S}_n) \rightarrow \mathbb{N}^n$ that if $\lambda(R) = (a_1, \dots, a_n)$, then $d(R_0, R) = a_1 + \dots + a_n$. Now use Theorem 1.3. \square

NOTE. An alternative proof of Corollary 1.2 is given by Athanasisdis [3].

6.4. The distance enumerator of a supersolvable arrangement

The goal of this section is a formula for the distance enumerator of a supersolvable (central) arrangement with respect to a “canonical” base region R_0 . The proof will be by induction, based on the following lemma of Björner, Edelman, and Ziegler [8].

Lemma 6.7. *Every central arrangement of rank 2 is supersolvable. A central arrangement \mathcal{A} of rank $d \geq 3$ is supersolvable if and only if $\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1$ (disjoint union), where \mathcal{A}_0 is supersolvable of rank $d-1$ (so $\mathcal{A}_1 \neq \emptyset$) and for all $H', H'' \in \mathcal{A}_1$ with $H' \neq H''$, there exists $H \in \mathcal{A}_0$ such that $H' \cap H'' \subseteq H$.*

Proof. Every geometric lattice of rank 2 is modular, hence supersolvable, so let \mathcal{A} be supersolvable of rank $d \geq 3$. Let $\hat{0} = x_0 < x_1 < \dots < x_{d-1} < x_d = \hat{1}$ be a modular maximal chain in $L_{\mathcal{A}}$. Define

$$\mathcal{A}_0 = \mathcal{A}_{x_{d-1}} = \{H \in \mathcal{A} : x_{d-1} \subseteq H\},$$

so $L(\mathcal{A}_0) \cong [\hat{0}, x_{d+1}]$. Clearly \mathcal{A}_0 is supersolvable of rank $d-1$. Let $\mathcal{A}_1 = \mathcal{A} - \mathcal{A}_0$. Let $H', H'' \in \mathcal{A}_1$, $H' \neq H''$. Since $x_{d-1} \not\subseteq H'$ we have $x_{d-1} \vee (H' \vee H'') = \hat{1}$ in $L(\mathcal{A})$. Now $\text{rk}(x_{d-1}) = d-1$, and $\text{rk}(H' \vee H'') = 2$ by semimodularity. Since x_{d-1} is modular we obtain

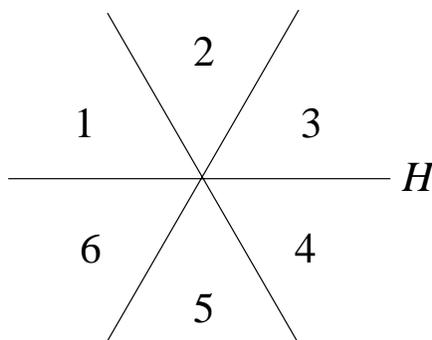
$$\text{rk}(x_{d-1} \wedge (H' \vee H'')) = (d-1) + 2 - d = 1,$$

i.e., $x_{d-1} \wedge (H' \vee H'') = H \in \mathcal{A}$. Since $H \leq x_{d-1}$ it follows that $H \in \mathcal{A}_0$. Moreover, $H' \cap H'' \subseteq H$ since $H \leq H' \vee H''$. This proves the “only if” part of the lemma. The “if” part is straightforward and not needed here, so we omit the proof. \square

Given $\mathcal{A}_0 = \mathcal{A}_{x_{d-1}}$ as above, define a map $\pi : \mathcal{R}(\mathcal{A}) \twoheadrightarrow \mathcal{R}(\mathcal{A}_0)$ (the symbol \twoheadrightarrow denotes surjectivity) by $\pi(R) = R'$ if $R \subseteq R'$. For $R \in \mathcal{R}(\mathcal{A})$ let

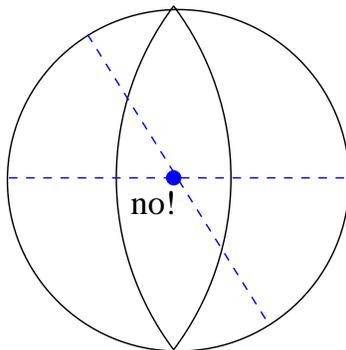
$$\mathcal{F}(R) = \{R_1 \in \mathcal{R}(\mathcal{A}) : \pi(R) = \pi(R_1)\} = \pi^{-1}(\pi(R)).$$

For example, let \mathcal{A} be the arrangement

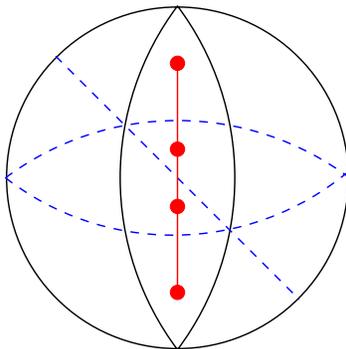


Let $\mathcal{A}_0 = \{H\}$. Then $\mathcal{F}(1) = \{1, 2, 3\}$ and $\mathcal{F}(5) = \{4, 5, 6\}$.

Now let $R' \in \mathcal{R}(\mathcal{A}_0)$. By Lemma 1.1 no $H', H'' \in \mathcal{A}$ can intersect inside R' . The illustration below is a projective diagram of a bad intersection. The solid lines define \mathcal{A}_0 and the dashed lines \mathcal{A}_1 .



Thus $\pi^{-1}(R')$ must be arranged “linearly” in R' , i.e., there is a straight line intersecting all $R \in \pi^{-1}(R')$.



Since $\text{rank}(\mathcal{A}) > \text{rank}(\mathcal{A}_0)$, we have $\#\pi^{-1}(R') > 1$ (for $H \in \mathcal{A}$ does not bisect R' if and only if $\text{rank}(\mathcal{A}_0 \cup H) = \text{rank}(\mathcal{A}_0)$). Thus there are two distinct regions $R_1, R_2 \in \pi^{-1}(R')$ that are endpoints of the “chain of regions.”

Let e_d have the meaning of equation (34), i.e.,

$$e_d = \#\{H \in \mathcal{A} : H \notin \mathcal{A}_0\} = \#\mathcal{A}_1.$$

Then $\pi^{-1}(R')$ is a chain of regions of length e_d , so $\#\pi^{-1}(R') = 1 + e_d$. We now come to the key definition of this subsection. The definition is recursive by rank, the base case being rank at most 2.

Definition 6.16. Let \mathcal{A} be a real supersolvable central arrangement of rank d , and let \mathcal{A}_0 be a supersolvable subarrangement of rank $d - 1$ (which always exists by the definition of supersolvability). A region $R_0 \in \mathcal{R}(\mathcal{A})$ is called *canonical* if either (1) $d \leq 2$, or else (2) $d \geq 3$, $\pi(R_0) \in \mathcal{R}(\mathcal{A}_0)$ is canonical, and R_0 is an endpoint of the chain $\mathcal{F}(R_0)$.

Since every chain has two endpoints and a central arrangement of rank 1 has two (canonical) regions, it follows that there are at least 2^d canonical regions.

The main result on distance enumerators of supersolvable arrangements is the following, due to Björner, Edelman, and Ziegler [8, Thm. 6.11].

Theorem 6.24. *Let \mathcal{A} be a supersolvable central arrangement of rank d in \mathbb{R}^n . Let $R_0 \in \mathcal{R}(\mathcal{A})$ be canonical, and suppose that*

$$\chi_{\mathcal{A}}(t) = (t - e_1)(t - e_2) \cdots (t - e_d)t^{n-d}.$$

(There always exist such positive integers e_i by Corollary 4.9.) Then

$$D_{\mathcal{A}, R_0}(t) = \prod_{i=1}^d (1 + t + t^2 + \cdots + t^{e_i}).$$

Proof. Let $W_{\mathcal{A}}$ be the weak order on \mathcal{A} with respect to R_0 , i.e.,

$$W_{\mathcal{A}} = \{\text{sep}(R_0, R) : R \in \mathcal{R}(\mathcal{A})\},$$

ordered by inclusion. Thus $W_{\mathcal{A}}$ is graded with rank function given by $\text{rk}(R) = d(R_0, R)$ and rank generating function

$$\sum_{R \in W_{\mathcal{A}}} t^{\text{rk}(R)} = D_{\mathcal{A}}(t).$$

Since R_0 is canonical, for all $R' \in \mathcal{R}(\mathcal{A}_0)$ we have that $\pi^{-1}(R')$ is a chain of length e_d . Hence if $R \in \mathcal{R}(\mathcal{A})$ and $h(R)$ denotes the rank of R in the chain $\mathcal{F}(R)$, then

$$d_{\mathcal{A}}(R_0, R) = d_{\mathcal{A}_0}(\pi(R)) + h(R).$$

Hence

$$D_{\mathcal{A}}(t) = D_{\mathcal{A}_0}(t)(1 + t + \cdots + t^{e_d}),$$

and the proof follows by induction. □

NOTE. The following two results were also proved in [8]. We simply state them here without proof.

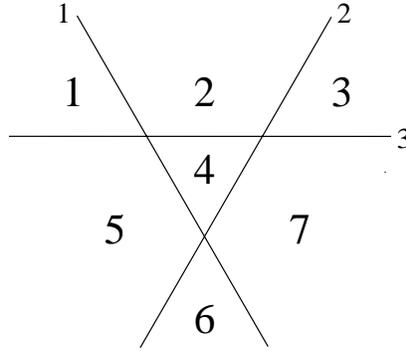
- If \mathcal{A} is a real supersolvable central arrangement and R_0 is canonical, then $W_{\mathcal{A}}$ is a lattice (Exercise 7).
- If \mathcal{A} is any real central arrangement and $W_{\mathcal{A}}$ is a lattice, then R_0 is simplicial (bounded by exactly $\text{rk}(\mathcal{A})$ hyperplanes, the minimum possible). In other words, the closure \bar{R}_0 is a simplex. As a partial converse, if every region R is simplicial, then $W_{\mathcal{A}}$ is a lattice (Exercise 8).

6.5. The Varchenko matrix

Let \mathcal{A} be a real arrangement. For each $H \in \mathcal{A}$ let a_H be an indeterminate. Define a matrix $V = V(\mathcal{A})$ with rows and columns indexed by $\mathcal{R}(\mathcal{A})$ by

$$V_{RR'} = \prod_{H \in \text{sep}(R, R')} a_H.$$

For instance, let \mathcal{A} be given as follows:



Then

	1	2	3	4	5	6	7
1	1	a_1	$a_1 a_2$	$a_1 a_3$	a_3	$a_2 a_3$	$a_1 a_2 a_3$
2	a_1	1	a_2	a_3	$a_1 a_3$	$a_1 a_2 a_3$	$a_2 a_3$
3	$a_1 a_2$	a_2	1	$a_2 a_3$	$a_1 a_2 a_3$	$a_1 a_3$	a_3
4	$a_1 a_3$	a_3	$a_2 a_3$	1	a_1	$a_1 a_2$	a_2
5	a_3	$a_1 a_3$	$a_1 a_2 a_3$	a_1	1	a_2	$a_1 a_2$
6	$a_2 a_3$	$a_1 a_2 a_3$	$a_1 a_3$	$a_1 a_2$	a_2	1	a_1
7	$a_1 a_2 a_3$	$a_1 a_3$	a_3	a_2	$a_1 a_2$	a_1	1

The determinant of this matrix happens to be given by

$$\det(V) = (1 - a_1^2)^3 (1 - a_2^2)^3 (1 - a_3^2)^3.$$

In order to state the general result, define for $x \in L(\mathcal{A})$,

$$\begin{aligned} a_x &= \prod_{H \supseteq x} a_H \\ n(x) &= r(\mathcal{A}^x) \\ p(x) &= b(c^{-1}\mathcal{A}_x) = \beta(\mathcal{A}_x), \end{aligned}$$

where as usual $\mathcal{A}^x = \{x \cap H \neq \emptyset : x \not\subseteq H\}$ and $\mathcal{A}_x = \{H \in \mathcal{A} : H \supseteq x\}$, and where c^{-1} denotes deconing and β is defined in Exercise 4.21. Thus

$$\begin{aligned} n(x) &= |\chi_{\mathcal{A}^x}(-1)| = \sum_{y \geq x} |\mu(x, y)| \\ p(x) &= |\chi'_{\mathcal{A}_x}(1)|. \end{aligned}$$

Example 6.14. The arrangement of three lines illustrated above has two types of intersections (other than $\hat{0}$): a line x and a point y . For a line x , \mathcal{A}^x consists of two points on a line, so $n(x) = r(\mathcal{A}^x) = 3$. Moreover, \mathcal{A}_x consists of the single hyperplane x in \mathbb{R}^2 , so $c^{-1}\mathcal{A}_x = \emptyset$ and $p(x) = b(\emptyset) = 1$. Hence we obtain the factor $(1 - a_x)^3$ in the determinant. On the other hand, $\mathcal{A}^y = \emptyset$ so $n(y) = r(\emptyset) = 1$. Moreover, \mathcal{A}_y consists of two intersecting lines in \mathbb{R}^2 , with characteristic polynomial $\chi_{\mathcal{A}_y}(t) = (t - 1)^2$. Hence $p(y) = |\chi'_{\mathcal{A}_y}(1)| = 0$. Equivalently, $c^{-1}\mathcal{A}_y$ consists of a single point on a line, so again $p(y) = b(c^{-1}\mathcal{A}_y) = 0$. Thus y contributes a factor $(1 - a_y^2)^0 = 1$ to $\det(V)$.

We can now state the remarkable result of Varchenko [37], generalized to “weighted matroids” by Brylawski and Varchenko [11].

Theorem 6.25. *Let \mathcal{A} be a real arrangement. Then*

$$\det V(\mathcal{A}) = \prod_{\hat{0} \neq x \in L(\mathcal{A})} (1 - a_x^2)^{n(x)p(x)}.$$

Proof. HELP! Can any reader explain to me the proof of Varchenko or Brylawski-Varchenko, or write it up comprehensibly? Incidentally, there are lots of interesting open problems in this area, discussed in G. Denham and P. Hanlon, Some algebraic properties of the Schechtman-Varchenko bilinear forms, in *New Perspectives in Algebraic Combinatorics*, MSRI Publ. 38, 1999, pp. 149–176, downloadable from <http://www.msri.org/publications/books/Book38/contents.html>.