Chapter 5

Synthetic-aperture radar

The object of synthetic aperture radar imaging (SAR) is to infer reflectivity profiles from measurement of scattered electromagnetic waves. The word "aperture" refers to the perceived angular resolution from the viewpoint of the sensor (antenna). The expression "synthetic aperture" refers to the fact that the aperture is created not from a very directional antenna, or array of antennas (as in ultrasound), but results from a computational process of triangulation, implicit in the handling of data with a backprojection formula.

The goal of the chapter is to gain an understanding of the geometry underlying the operators F and F^* arising in SAR. Our reference for this chapter is the book "Fundamentals of radar imaging" by Cheney and Borden.

5.1 Assumptions and vocabulary

We will make the following basic assumptions: (!)

1. *Scalar fields* obeying the wave equation, rather than vector fields obeying Maxwell's equation. This disregards polarization (though processing polarization is a sometimes a simple process of addition of images.) The reflectivity of the scatterers is then encoded via $m(x)$ as usual, rather than by specifying the shape of the boundary $\partial\Omega$ and the type of boundary conditions for the exterior Maxwell problem.

2. *The Born approximation*, so that data d are proportional to εu_1, and $u_1 = Fm_1$. This disregards multiple scattering. In the sequel we will write $\varepsilon = 1$ for simplicity.

3. *No dispersion*, so that all waves travel at the same speed regardless of frequency, as in the wave equation. Dispersion happens for radio waves in the ionosphere.

4. The reflectivity $m(x) = m_0(x) + \varepsilon m_1(x)$ is *constant in time*, with m_0 *constant in time and space*. This disregards moving scatterers. As mentioned earlier, we put $\varepsilon = 1$. For convenience, we will also drop the subscript 1 from m_1, so that *in this chapter, m stands for the perturbation* in squared slowness $1/c^2$.

(!) A few other "working" assumptions are occasionally made for convenience, but can easily be removed if necessary:

5. *The far field assumption*: spherical wavefronts are assumed to be locally planar, for waves at the scatterer originating from the antenna (or vice-versa).

6. *Monostatic SAR*: the same antenna is used for transmission and reception. It is not difficult to treat the bistatic/multistatic case where different antennas play different roles.

7. *Start-stop approximation*: in the time it takes for the pulse to travel back and forth from the antenna to the scatterers, the antenna is assumed not to have moved.

8. *Flat topography*: the scatterers are located at elevation $z = 0$.

SAR typically operates with radio waves or microwaves, with wavelengths on the order of meters to centimeters. Moving antennas are typically carried by planes or satellites. A variant of SAR is to use arrays of fixed antennas, a situation called MIMO (multiple input, multiple output.) If the frequency band is of the form $[\omega_0 - \Delta\omega/2, \omega_0 + \Delta\omega/2]$, we say ω_0 is the *carrier frequency* and $\Delta\omega$ is the *bandwidth*. We speak of *wideband* acquisition when $\Delta\omega$ is a large fraction of ω_0. As usual, $\omega = 2\pi\nu$ where ν is in Hertz.

The direction parallel to the trajectory of the antenna is called *along-track*. The vector from the antenna to the scatterer is called *range vector*, its direction is the *range direction*, and the direction perpendicular to the range direction is called *cross-range*. The distance from the antenna to the scatterer is also called *range*. The length of the horizontal projection of the range vector is the *downrange*.

We will not deal with the very interesting topic of Doppler imaging, where frequency shifts are used to infer velocities of scatterers. We will also not (!) cover the important topic of interferometric SAR (InSAR) where the objective is to create difference images from time-lapse datasets.

We finish this section by describing the nature of the far-field approximation in more details, and its consequence for the expression of the Green's function $\frac{e^{ik|x-y|}}{4\pi|x-y|}$. Consider an antenna located near the origin. We will assume that a scatterer at x is "far" from a point y on the antenna in the sense that

$$|y| \ll |x|, \qquad k|y|^2 \ll |x|.$$

Then, if we let $\widehat{x} = \frac{x}{|x|}$,

$$\begin{aligned}
|x-y| &= \sqrt{|x|^2 - 2x\cdot y + |y|^2} \\
&= |x|\sqrt{1 - 2\frac{\widehat{x}\cdot y}{|x|} + \frac{|y|^2}{|x|^2}} \\
&\simeq |x|\left(1 - \frac{\widehat{x}\cdot y}{|x|} + \frac{1}{2}\frac{|y|^2}{|x|^2} + \ldots\right) \\
&= |x| - \widehat{x}\cdot y + \frac{1}{2}\frac{|y|^2}{|x|} + \ldots
\end{aligned}$$

We therefore have

$$e^{ik|x-y|} = e^{ik|x|}e^{-ik\widehat{x}\cdot y}\left(1 + O\left(\frac{k|y|^2}{|x|}\right)\right),$$

$$\frac{1}{|x-y|} = \frac{1}{|x|}\left(1 + O\left(\frac{|y|}{|x|}\right)\right).$$

As a result, in the far field,

$$\frac{e^{ik|x-y|}}{4\pi|x-y|} \simeq \frac{e^{ik|x|}}{4\pi|x|}e^{-ik\widehat{x}\cdot y}.$$

This simplification will cause the y integrals to become Fourier transforms.

5.2 Forward model

We can now inspect the *radiation field* created by the antenna at the transmission side. The $\simeq$ sign will be dropped for $=$, although it is understood that the approximation is only accurate in the far field. Call $j(x,\omega)$ the scalar analogue of the vector forcing generated by currents at the antenna, called current density vector. (The dependence on ω is secondary.) Call $\widehat{p}(\omega)$ the Fourier transform of the user-specified pulse $p(t)$. Then

$$\widehat{u_0}(x,\omega) = \int \frac{e^{ik|x|}}{4\pi|x|} e^{-ik\widehat{x}\cdot y} j(y,\omega)\widehat{p}(\omega)\, dy.$$

This reduces to a spatial Fourier transform of j in its first argument,

$$\widehat{u_0}(x,\omega) = \frac{e^{ik|x|}}{4\pi|x|} \widehat{j}^{(1)}(k\widehat{x},\omega)\widehat{p}(\omega).$$

For short, we let

$$J(\widehat{x},\omega) = \widehat{j}^{(1)}(k\widehat{x},\omega),$$

and call it the *radiation beam pattern*. It is determined by the shape of the antenna. As a function of $\widehat{x}$, the radiation beam pattern is often quite broad (not concentrated).

For an antenna centered at position $\gamma(s)$, parametrized by s (called slow time), the radiation field is therefore

$$\widehat{u_{0,s}}(x,\omega) = \frac{e^{ik|x-\gamma(s)|}}{4\pi|x-\gamma(s)|} J(\widehat{x-\gamma(s)},\omega)\widehat{p}(\omega).$$

The scattered field $u_1(x,\omega)$ is not directly observed. Instead, the recorded data are the linear functionals

$$\widehat{d}(s,\omega) = \int_{A_s} u_1(y,\omega) w(y,\omega)\, dy$$

against some window function $w(x,\omega)$, and where the integral is over the antenna A_s centered at $\gamma(s)$. Recall that u_1 obeys (4.5), hence (with m standing for what we used to call m_1)

$$\widehat{d}(s,\omega) = \int_{A_s} \int \frac{e^{ik|x-y|}}{4\pi|x-y|} \omega^2 \widehat{u_0}(x,\omega) m(x) w(y,\omega)\, dydx.$$

In the regime of the the far-field approximation for an antenna at $\gamma(s)$, we get instead (still using an equality sign)

$$\widehat{d}(s,\omega) = \int \frac{e^{ik|x-\gamma(s)|}}{4\pi|x-\gamma(s)|} \omega^2 \widehat{u_0}(x,\omega) m(x) \widehat{w}^{(1)}(k(\widehat{x-\gamma(s)}),\omega).$$

The start-stop approximation results in the same $\gamma(s)$ used at transmission and at reception. For short, we let

$$W(\widehat{x}, \omega) = \widehat{w}^{(1)}(k\widehat{x}, \omega),$$

and call it the *reception beam pattern*. For a perfectly conducting antenna, the two beam patterns are equal by reciprocity: ($)

$$J(\widehat{x}, \omega) = W(\widehat{x}, \omega).$$

We can now carry through the substitutions and obtain the expression of the linearized forward model F:

$$\widehat{d}(s, \omega) = \widehat{Fm}(s, \omega) = \int e^{2ik|x-\gamma(s)|} A(x, s, \omega) m(x)\, dx, \tag{5.1}$$

with amplitude

$$A(x, s, \omega) = \omega^2\, \widehat{p}(\omega)\, \frac{J(\widehat{x-\gamma(s)}, \omega) W(\widehat{x-\gamma(s)}, \omega)}{16\pi^2 |x-\gamma(s)|^2}.$$

So far we have assumed that $x = (x_1, x_2, x_3)$, and that dx a volume element. We could alternatively assume a two-dimensional reflectivity profile at a known elevation $x_3 = h(x_1, x_2)$. In that case we write

$$x_T = (x_1, x_2, h(x_1, x_2)),$$

assume a reflectivity of the form $m(x) = \delta(x_3 - h(x_1, x_2)) V(x_1, x_2)$, and get (!)

$$\widehat{d}(s, \omega) = \int e^{2ik|x_T-\gamma(s)|} A(x_T, s, \omega) V(x_1, x_2)\, dx_1 dx_2.$$

In the sequel we assume $h = 0$ for simplicity. We also abuse notations slightly (!)
and write $A(x, s, \omega)$ for the amplitude.

The geometry of the formula for F is apparent if we return to the time variable. For illustration, reduce $A(x, s, \omega) = \omega^2 \widehat{p}(\omega)$ to its leading ω dependence. Then

$$\begin{aligned} d(s, t) &= \frac{1}{2\pi} \int e^{-i\omega t} \widehat{d}(s, \omega)\, d\omega \\ &= -\frac{1}{2\pi} \int p''\left(t - 2\frac{|x-\gamma(s)|}{c_0} \right) m(x)\, dx. \end{aligned}$$

We have used the fact that $k = \omega/c_0$ to help reduce the phase to the simple expression

$$t - 2\frac{|x - \gamma(s)|}{c}$$

Its physical significance is clear: the time taken for the waves to travel to the scatterer and back is twice the distance $|x - \gamma(s)|$ divided by the light speed c_0. Further assuming $p(t) = \delta(t)$, then there will be signal in the data $d(s,t)$ only at a time $t = 2\frac{|x-\gamma(s)|}{c}$ compatible with the kinematics of wave propagation. The locus of possible scatterers giving rise to data $d(s,t)$ is then a sphere of radius $ct/2$, centered at the antenna $\gamma(s)$. It is a good exercise to modify these conclusions in case $p(t)$ is a narrow pulse (oscillatory bump) supported near $t = 0$, or even when the amplitude is returned to its original form with beam patterns.

In SAR, s is called slow time, t is the fast time, and as we mentioned earlier, $|x - \gamma(s)|$ is called range.

5.3 Filtered backprojection

In the setting of the assumptions of section 5.1, the imaging operator F^* is called *backprojection* in SAR. Consider the data inner product[1]

$$\langle d, Fm \rangle = \int \widehat{d}(s,\omega)\overline{\widehat{Fm}(s,\omega)}\, ds d\omega.$$

As usual, we wish to isolate the dependence on m to identify $\langle d, Fm \rangle$ as $\langle F^* d, m \rangle$. After using (5.1), we get

$$\langle d, Fm \rangle = \int m(x) \iint e^{-2ik|x-\gamma(s)|}\overline{A(x,s,\omega)}\widehat{d}(s,\omega)\, ds d\omega\, dx.$$

This means that

$$(F^* d)(x) = \iint e^{-2ik|x-\gamma(s)|}\overline{A(x,s,\omega)}\widehat{d}(s,\omega)\, ds d\omega. \tag{5.2}$$

Notice that the kernel of F^* is the *conjugate* of that of F, and that the integration is over the data variables (s,ω) rather than the model variable x.

[1] It could be handy to introduce a multiplicative factor 2π in case the Parseval identity were to be used later.

The physical interpretation is clear if we pass to the t variable, by using $\widehat{d}(s,\omega) = \int e^{i\omega t} d(s,t)\,dt$ in (5.2). Again, assume $A(x,s,\omega) = \omega^2 \widehat{p}(\omega)$. We then have

$$(F^* d)(x) = -\frac{1}{2\pi} \int p''\left(t - 2\frac{|x-\gamma(s)|}{c_0} \right) d(s,t)\, ds dt.$$

Assume for the moment that $p(t) = \delta(t)$; then F^* places a contribution to the reflectivity at x if and only if there is signal in the data $d(s,t)$ for s,t,x linked by the same kinematic relation as earlier, namely $t = 2\frac{|x-\gamma(s)|}{c}$. In other words, it "spreads" the data $d(s,t)$ along a sphere of radius $ct/2$, centered at $\gamma(s)$, and adds up those contributions over s and t. In practice p is a narrow pulse, not a delta, hence those spheres become thin shells. Strictly speaking, "backprojection" refers to the amplitude-free formulation $A =$ constant, i.e., in the case when $p''(t) = \delta(t)$. But we will use the word quite liberally, and still refer to the more general formula (5.2) as backprojection. So do many references in the literature.

Backprojection can also be written in the case when the reflectivity profile is located at elevation $h(x_1,x_2)$. It suffices to evaluate (5.2) at $x_T = (x_1, x_2, h(x_1,x_2))$.

We now turn to the problem of modifying backprojection to give a formula approximating F^{-1} rather than F^*. Hence the name *filtered backprojection*. It will only be an approximation of F^{-1} because of sampling issues, as we will see.

The phase $-2ik|x-\gamma(s)|$ needs no modification: it is already "kinematically correct" (for deep reasons that will be expanded on at length in the chapter on microlocal analysis). Only the amplitude needs to be changed, to yield a new operator[2] B to replace F^*:

$$(Bd)(x) = \iint e^{-2ik|x-\gamma(s)|} Q(x,s,\omega) \widehat{d}(s,\omega)\, ds d\omega.$$

By composing B with F, we obtain

$$(BFm)(x) = \int K(x,y) m(y)\, dy,$$

[2] B for filtered Backprojection, or for Gregory Beylkin, who was the first to propose it in 1984.

with

$$K(x,y) = \iint_{\mathcal{M}} e^{-2ik|x-\gamma(s)|+2ik|y-\gamma(s)|} Q(x,s,\omega) A(y,s,\omega)\, ds d\omega. \tag{5.3}$$

The integral runs over the so-called data manifold $\mathcal{M}$. We wish to choose Q so that BF is as close to the identity as possible, i.e.,

$$K(x,y) \simeq \delta(x-y).$$

This can be done by reducing the oscillatory integral in (5.3) to an integral of the form

$$\frac{1}{(2\pi)^2} \int e^{i(x-y)\cdot\xi}\, d\xi,$$

which, as we know, equals $\delta(y-x)$ if the integral is taken over $\mathbb{R}^2$. The integral will turn out to be over a bounded set, the characterization of which is linked to the question of resolution as explained in the next section, but the heuristic that we want to approach $\delta(y-x)$ remains relevant.

As the integral in (5.3) is in data space (s,ω), we define $\xi \in \mathbb{R}^2$ as the result of an as-yet undetermined change of variables

$$(s,\omega) \mapsto \xi = \Xi(s,\omega;x).$$

(ξ is xi, Ξ is capital xi.) The additional dependence on x indicates that the change of variables will be different for each x.

To find Ξ, we need to introduce some notations. We follow Borden-Cheney [**?**] closely. Denote the range vector by

$$R_{y,s} = \gamma(s) - y_T$$

For reference, its partials are

$$\frac{\partial R_{y,s}}{\partial s} = \dot\gamma(s),$$

$$\nabla_y R_{y,s} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = -P_2.$$

We understand both $R_{y,s}$ and $\frac{\partial R_{y,s}}{\partial s}$ a column 3-vectors in a matrix context. The modification to deal with a nonzero elevation $h(x_1,x_2)$ in x_T is simple. Then

$$\frac{\partial}{\partial s}|R_{y,s}| = \left(\frac{\partial R_{y,s}}{\partial s}\right)^T \frac{R_{y,s}}{|R_{y,s}|} = \dot\gamma(s)\cdot\widehat{R_{y,s}},$$

$$\nabla_y |R_{y,s}| = \left(\frac{\partial R_{y,s}}{\partial y} \right)^T \frac{R_{y,s}}{|R_{y,s}|} = P_2^T \, \widehat{R_{y,s}},$$

where $\widehat{R_{y,s}}$ is the unit range vector. The operation of pre-multiplying a column 3-vector by P_2^T corresponds to extraction of the first two components of the vector. (Recall that x and y are coordinates in two dimensions, while their physical realizations x_T and y_T have a zero third component.)

With the partial derivatives in hand we can now apply the principle of stationary phase (see appendix C) to the integral (5.3). The coordinates x and y are fixed when considering the phase

$$\phi(s, \omega) = 2k(|R_{y,s}| - |R_{x.s}|).$$

We can introduce a large parameter α in the phase by normalizing frequencies as $\omega = \alpha\omega'$ (recall $k = \omega/c$); the higher the frequency band of the pulse the better the approximation from stationary phase asymptotics. The critical points occur when

$$\frac{\partial \phi}{\partial \omega} = \frac{2}{c}(|R_{y,s}| - |R_{x.s}|) = 0,$$

$$\frac{\partial \phi}{\partial s} = 2k \, \dot{\gamma}(s) \cdot (\widehat{R_{y,s}} - \widehat{R_{x,s}}) = 0.$$

The Hessian matrix is singular, which seemingly precludes a direct application of lemma 5 in appendix 4, but the second example following the lemma shows the trick needed to remedy the situation: use a trial function $f(y)$ and extend the integration variables to also include y. Henceforth we denote the phase as $\phi(s, \omega, y)$ to stress the extra dependence on y.

The critical points occur when 1) the *ranges* are equal, and 2) the *downrange velocities* are equal. For fixed x, the first condition can be visualized in three-dimensional y_T-space as a sphere centered about $\gamma(s)$, and passing through x_T. The second condition corresponds to a cone with symmetry axis along the tangent vector $\dot{\gamma}(s)$ to the trajectory, and with the precise opening angle that ensures that x_T belongs to the cone. Thirdly, we have $y_T = (y_1, y_2, 0)$, so an additional intersection with the horizontal plane $z = 0$ should be taken. The intersection of the sphere, the cone, and the plane, consists of two points: $y_T = x_T$, and $y_T = x_{T,\mathrm{mirr}}$, the mirror image of x_T about the local flight plane (the vertical plane containing $\dot{\gamma}(s)$). In practice, the antenna beam pattern "looks to one side", so that $A(x, s, \omega) \simeq 0$ for x on the "uninteresting" side of the flight path, therefore the presence of $x_{T,\mathrm{mirr}}$

can be ignored. (If not, the consequence would be that SAR images would be symmetric about the flight plane.)

With the critical point essentially unique and at $y = x$, we can invoke stationary phase to claim that the main contribution to the integral is due to points y near x. This allows to simplify the integral (5.3) in two ways: 1) the amplitude $A(y, s, \omega)$ is smooth enough in y that we can approximate it by $A(x, s, \omega)$, and 2) the phase factor can be approximated as locally linear in $y - x$, as

(!)

$$\phi(s, \omega, y) = 2k(|R_{y,s}| - |R_{x.s}|) \simeq (y - x) \cdot \xi.$$

A multivariable Taylor expansion reveals that ξ can be chosen as the y-gradient of the phase, evaluated at x:

$$\xi = \Xi(x, \omega; x) = \nabla_y \phi(s, \omega, y)|_{y=x} = 2kP_2^T \widehat{R_{x,s}}.$$

We have therefore reduced the expression of $K(y, x)$ to

$$K(x, y) \simeq \int_{\mathcal{M}} e^{i(y-x)\cdot\Xi(s,\omega;x)} Q(x, s, \omega) A(x, s, \omega) \, ds d\omega.$$

Changing from (s, ω) to ξ variables, and with a reasonable abuse of notation in the arguments of the amplitudes, we get

$$K(x, y) \simeq \int e^{i(y-x)\cdot\xi} Q(x, \xi) A(x, \xi) \, |\frac{\partial(s, \omega)}{\partial \xi}| \, d\xi.$$

The Jacobian $J = |\frac{\partial(s,\omega)}{\partial \xi}|$ of the change of variables goes by the name *Beylkin determinant*.

The proper choise of Q that will make this integral close to $\int e^{i(y-x)\cdot\xi} \, d\xi$ is now clear: we should take

$$Q(x, \xi) = \frac{1}{A(x, \xi) |\frac{\partial(s,\omega)}{\partial \xi}|} \chi(x, \xi), \tag{5.4}$$

(!) for some adequate cutoff $\chi(x, \xi)$ to prevent division by small numbers. The presence of χ owes partly to the fact that A can be small, but also partly (and mostly) to the fact that the data variables (s, ω) are limited to the data manifold $\mathcal{M}$. The image of $\mathcal{M}$ in the ξ domain is now an x-dependent set that we may denote $\Xi(\mathcal{M}; x)$. The cutoff $\chi(x, \xi)$ essentially indicates this set in the ξ variable, in a smooth way so as to avoid unwanted ringing artifacts.

The conclusion is that, when Q is given by (5.4), the filtered backprojection operator B acts as an approximate inverse of F, and the kernel of BF is well modeled by the approximate identity

$$K(x,y) \simeq \int_{\Xi(\mathcal{M};x)} e^{i(y-x)\cdot\xi}\, d\xi.$$

5.4 Resolution

See Borden-Cheney chapter 9.

(...)

$$\Delta x_1 = \frac{c}{\Delta\omega \ \sin\psi}$$

$$\Delta x_2 = \frac{L}{2}, \qquad L \geq \lambda$$

5.5 Exercises

1. Prove (5.2) in an alternative fashion by substituting in the far-field approximation of G in the imaging condition (4.7).

2. *Bistatic SAR*: repeat and modify the derivation of (5.1) in the case of an antenna $\gamma_1(s)$ for transmission and another antenna $\gamma_2(s)$ for reception.

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