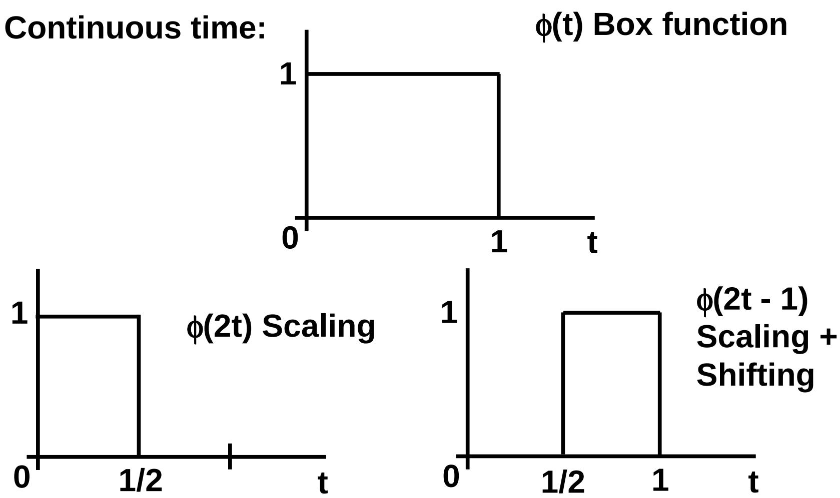
Course 18.327 and 1.130
Wavelets and Filter Banks

Multiresolution Analysis (MRA):
Requirements for MRA;
Nested Spaces and
Complementary Spaces;
Scaling Functions and Wavelets

Scaling Functions and Wavelets

Continuous time:



2

For this example:

$$\phi(t) = \phi(2t) + \phi(2t - 1)$$

More generally:

$$\boxed{\phi(t) = 2\sum_{k=0}^{N} h_0[k]\phi(2t - k)}$$

Refinement equation
or
Two-scale difference equation

$\phi(t)$ is called a scaling function

The refinement equation couples the representations of a continuous-time function at two time scales. The continuous-time function is determined by a discrete-time filter, $h_0[n]$! For the above (Haar) example:

$$h_0[0] = h_0[1] = ½ \quad \text{(a lowpass filter)}$$

3

Note: (i) Solution to refinement equation may not always exist. If it does…

(ii) $\phi(t)$ has compact support i.e.

$\phi(t) = 0$ outside $0 \leq t < N$

(comes from the FIR filter, $h_0[n]$)

(iii) $\phi(t)$ often has no closed form solution.

(iv) $\phi(t)$ is unlikely to be smooth.

Constraint on $h_0[n]$:

$$\int \phi(t)dt = 2\sum_{k=0}^{N} h_0[k] \int \phi(2t - k)dt$$

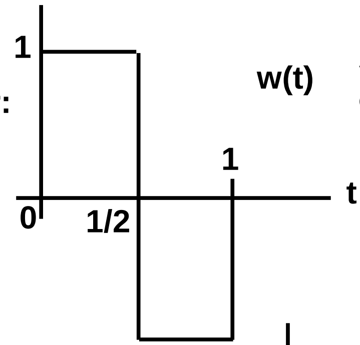
$$= 2\sum_{k=0}^{N} h_0[k] \bullet ½ \int \phi(\tau)d\tau$$

So

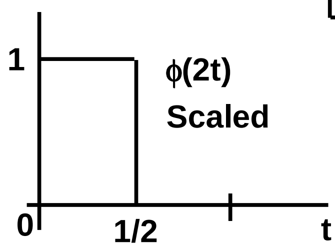
$$\boxed{\sum_{k=0}^{N} h_0[k] = 1} \quad \text{Assumes } \int \phi(t)dt \neq 0$$

4

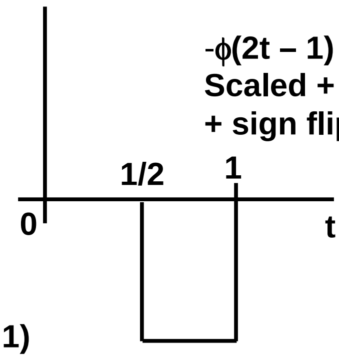
Now consider:



Square wave
of finite length -
Haar wavelet



$\phi(2t)$
Scaled



$-\phi(2t - 1)$
Scaled + shifted
+ sign flipped

$w(t) = \phi(2t) - \phi(2t - 1)$

5

More generally:

$$w(t) = 2\sum_{k=0}^{N} h_1[k]\, \phi(2t - k)$$

Wavelet equation

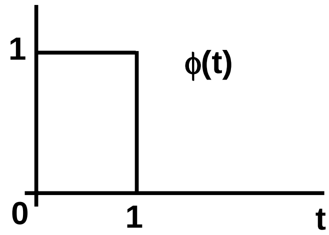
For the Haar wavelet example:

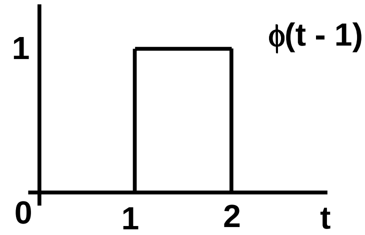
$h_1[0] = ½ \quad h_1[1] = -½$ (a highpass filter)

6

Some observations for Haar scaling function and wavelet

1. **Orthogonality of integer shifts (translates):**





$$\int \phi(t)\,\phi(t-k)dt = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \delta[k]$$

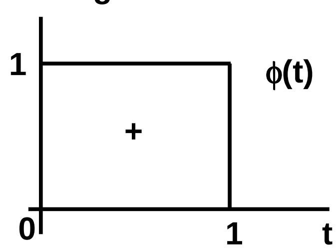
Similarly

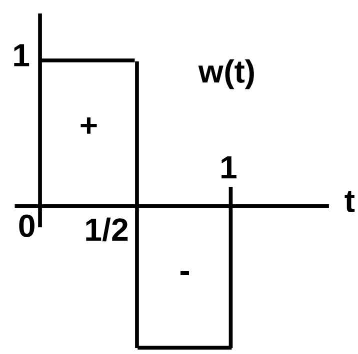
$$\int w(t)\,w(t-k)dt = \delta[k]$$

Reason: no overlap

7

2. **Scaling function is orthogonal to wavelet:**



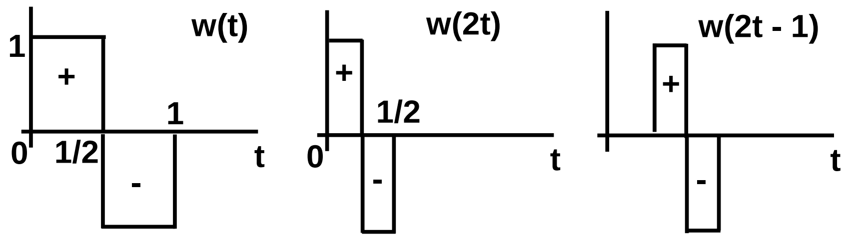


$$\int \phi(t)\,w(t)dt = 0$$

Reason: +ve and –ve areas cancel each other.

8

3. Wavelet is orthogonal across scales:



$\int$ w(t) w(2t)dt = 0 , $\int$ w(t) w(2t – 1)dt = 0

Reason: finer scale versions change sign while coarse scale version remains constant.

9

Wavelet Bases

Our goal is to use w(t), its scaled versions (dilations) and their shifts (translates) as building blocks for continuous-time functions, f(t). Specifically, we are interested in the class of functions for which we can define the inner product:

$$\langle f(t), g(t)\rangle = \int_{-\infty}^{\infty} f(t)\, g^*(t)\,dt < \infty$$

Such functions f(t) must have finite energy:

$$\|f(t)\|^2 = \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty$$

and they are said to belong to the Hilbert space, $L^2(\Re)$.

10

Consider all dilations and translates of the Haar wavelet:

$w_{j,k}(t) = 2^{j/2}\, w(2^j t - k)$; $-\infty \le j \le \infty$

$-\infty \le k \le \infty$

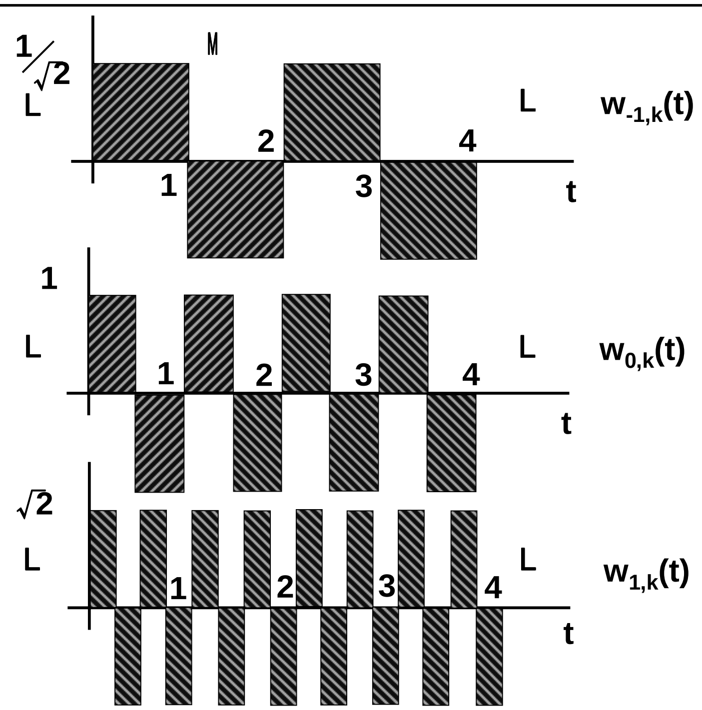
↑ Normalization factor so that $\|w_{j,k}(t)\| = 1$

$$\int w_{j,k}(t)\, w_{J,K}(t)\, dt = \int 2^{j/2}\, w(2^j t - k) \,.\, 2^{J/2}\, w(2^J t - K)\, dt$$

$$= \begin{cases} 1 \text{ if } j = J \text{ and } k = K \\ 0 \text{ otherwise} \end{cases}$$

$$= \delta[\, j - J \,]\, \delta[\, k - K \,]$$

11



12

$w_{jk}(t)$ form an orthonormal basis for $L^2(\Re)$.

$$f(t) = \sum_{j,k} b_{jk}\, w_{jk}(t) \;; \qquad w_{jk}(t) = 2^{j/2}\, w(2^j t - k)$$

$$b_{jk} = \int_{-\infty}^{\infty} f(t)\, w_{jk}(t)\, dt$$

13

Multiresolution Analysis

Key ingredients:

1. A sequence of embedded subspaces:

$$\{0\} \subset \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset L^2(\Re)$$

$L^2(\Re)$ = all functions with finite energy

$$= \{f(t): \int_{-\infty}^{\infty} |f(t)|^2\, dt < \infty\}$$ Hilbert space

Requirements:

- Completeness as $j \to \infty$. If $f(t)$ belongs to $L^2(\Re)$ and $f_j(t)$ is the portion of $f(t)$ that lies in V_j, then $\lim_{j\to\infty} f_j(t) = f(t)$

14

Restated as a condition on the subspaces:

$$\overline{\bigcup_{j=-\infty}^{\infty} V_j} = L^2(\Re)$$

- Emptiness as $j \to -\infty$

$$\lim_{j \to -\infty} \| f_j(t) \| = 0$$

Restated as a condition on the subspaces:

$$\bigcap_{j=-\infty}^{\infty} V_j = \{0\}$$

15

2. A sequence of complementary subspaces, W_j, such that $V_j + W_j = V_{j+1}$

   and $V_j \cap W_j = \{0\}$ (no overlap)

   This is written as

$$V_j \oplus W_j = V_{j+1} \text{ (Direct sum)}$$

Note: An orthogonal multiresolution will have W_j orthogonal to V_j : $W_j \perp V_j$.
So orthogonality will ensure that $V_j \cap W_j = \{0\}$

16

We thus have

$$V_1 = V_0 \oplus W_0$$
$$V_2 = V_1 \oplus W_1 = V_0 \oplus W_0 \oplus W_1$$
$$V_3 = V_2 \oplus W_2 = V_0 \oplus W_0 \oplus W_1 \oplus W_2$$
$$\vdots$$
$$V_J = V_{J-1} \oplus W_{J-1} = V_0 \oplus \sum_{j=0}^{J-1} W_j$$
$$\vdots$$
$$L^2(\Re) = V_0 \oplus \sum_{j=0}^{\infty} W_j$$

We can also write the recursion for j < 0

$$V_0 = V_{-1} \oplus W_{-1}$$
$$= V_{-2} \oplus W_{-2} \oplus W_{-1}$$
$$\vdots$$
$$= V_{-k} \oplus \sum_{j=-k}^{-1} W_j$$
$$\vdots$$
$$= \sum_{j=-\infty}^{-1} W_j \qquad \Rightarrow L^2(\Re) = \sum_{j=-\infty}^{\infty} W_j$$

17

3. A scaling (dilation) law:
   If $f(t) \in V_j$ then $f(2t) \in V_{j+1}$
4. A shift (translation) law:
   If $f(t) \in V_j$ then $f(t-k) \in V_j$ k integer
5. V_0 has a shift-invariant basis, $\{\phi(t-k) : -\infty \le k \le \infty\}$
   W_0 has a shift-invariant basis, $\{w(t-k) : -\infty \le k \le \infty\}$

We expect that $V_1 = V_0 + W_0$ will have twice as many basis functions as V_0 alone.

First possibility: $\{\phi(t-k), w(t-k) : -\infty \le k \le \infty\}$

Second possibility: use the scaling law i.e.

if $\phi(t-k) \in V_0$, then $\phi(2t-k) \in V_1$

18

So

V_1 has a shift-invariant basis, $\{\sqrt{2}\,\phi(2t-k): -\infty \leq k \leq \infty\}$

Can we relate this basis for V_1 to the basis for V_0?
We know that

$$V_0 \subset V_1$$

So any function in V_0 can be written as a combination of the basic functions for V_1.
In particular, since $\phi(t) \in V_0$, we can write

$$\phi(t) = 2\sum_k h_0[k]\,\phi(2t - k)$$

This is the Refinement Equation (a.k.a. the Two-Scale Difference Equation or the Dilation Equation).

19

We also know that

$$W_0 = V_1 - V_0$$

So

$$W_0 \subset V_1$$

This means that any function in W_0 can also be written as a combination of the basic functions for V_1.
Since $w(t) \in W_0$, we can write

$$w(t) = 2\sum_k h_1[k]\,\phi(2t - k)$$

Wavelet Equation

20

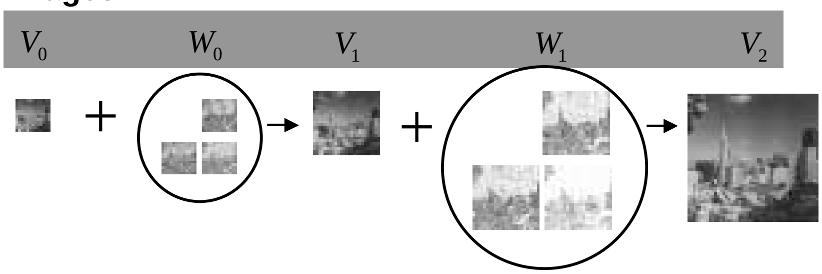
Multiresolution Representations

Functions:

$$L^2(\Re) = V_0 \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots$$

- Level 2 detail
- Level 1 detail
- Level 0 detail
- Finite energy functions
- Coarse approximation

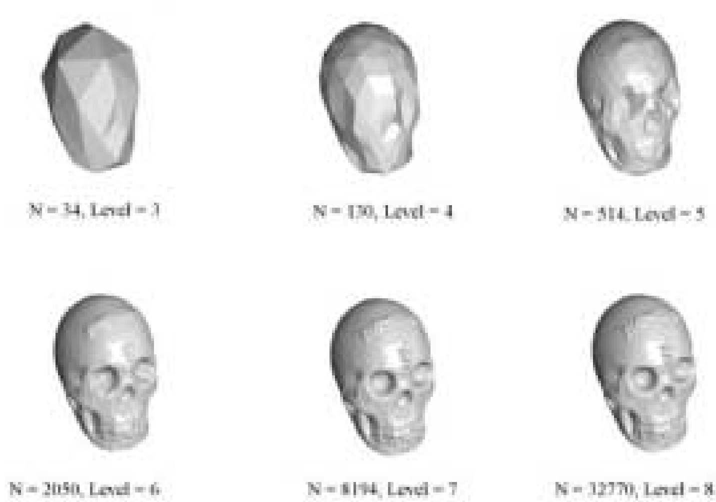
Images:



21

Multiresolution Representations

Geometry:



22

Mesh courtesy of Igor Guskov (Caltech)