Course 18.327 and 1.130
Wavelets and Filter Banks

Filter Banks (contd.): perfect reconstruction; halfband filters and possible factorizations.

Product Filter

Example: Product filter of degree 6

$$P_0(z) = \frac{1}{16}(-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6})$$

$$P_0(z) - P_0(-z) = 2z^{-3}$$

$\Rightarrow$ Expect perfect reconstruction with a 3 sample delay

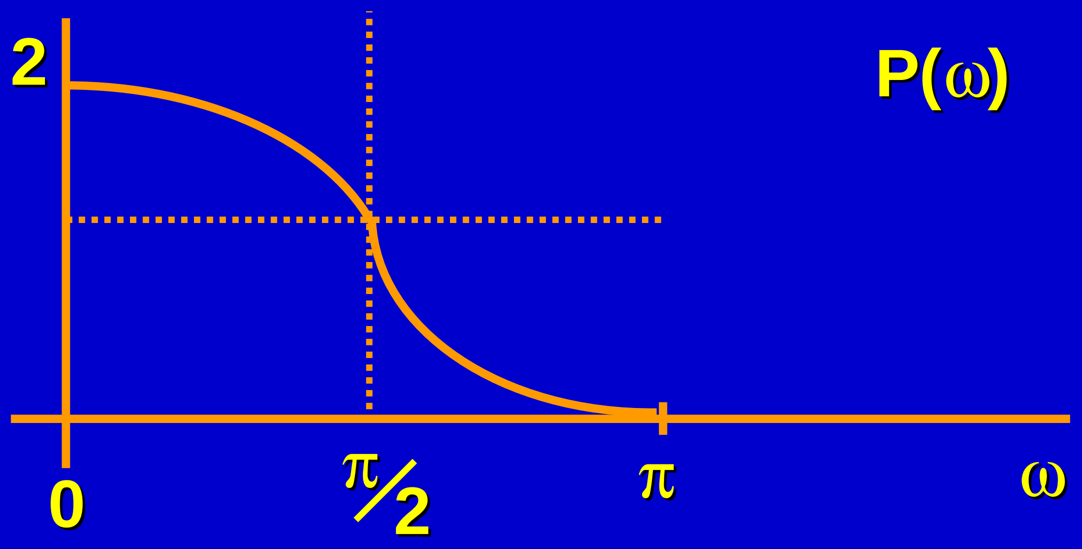
Centered form:

$$P(z) = z^3 P_0(z) = \frac{1}{16}(-z^3 + 9z + 16 + 9z^{-1} - z^{-3})$$

$P(z) + P(-z) = 2$ i.e. even part of $P(z)$ = const

In the frequency domain:

$$\boxed{P(\omega) + P(\omega + \pi) = 2}$$ Halfband Condition



Note antisymmetry about $\omega = \pi/2$

$P(\omega)$ is said to be a halfband filter.

How do we factor $P_0(z)$ into $H_0(z)\ F_0(z)$?

$$P_0(z) = 1/16(1 + z^{-1})^4(-1 + 4z^{-1} - z^{-2})$$

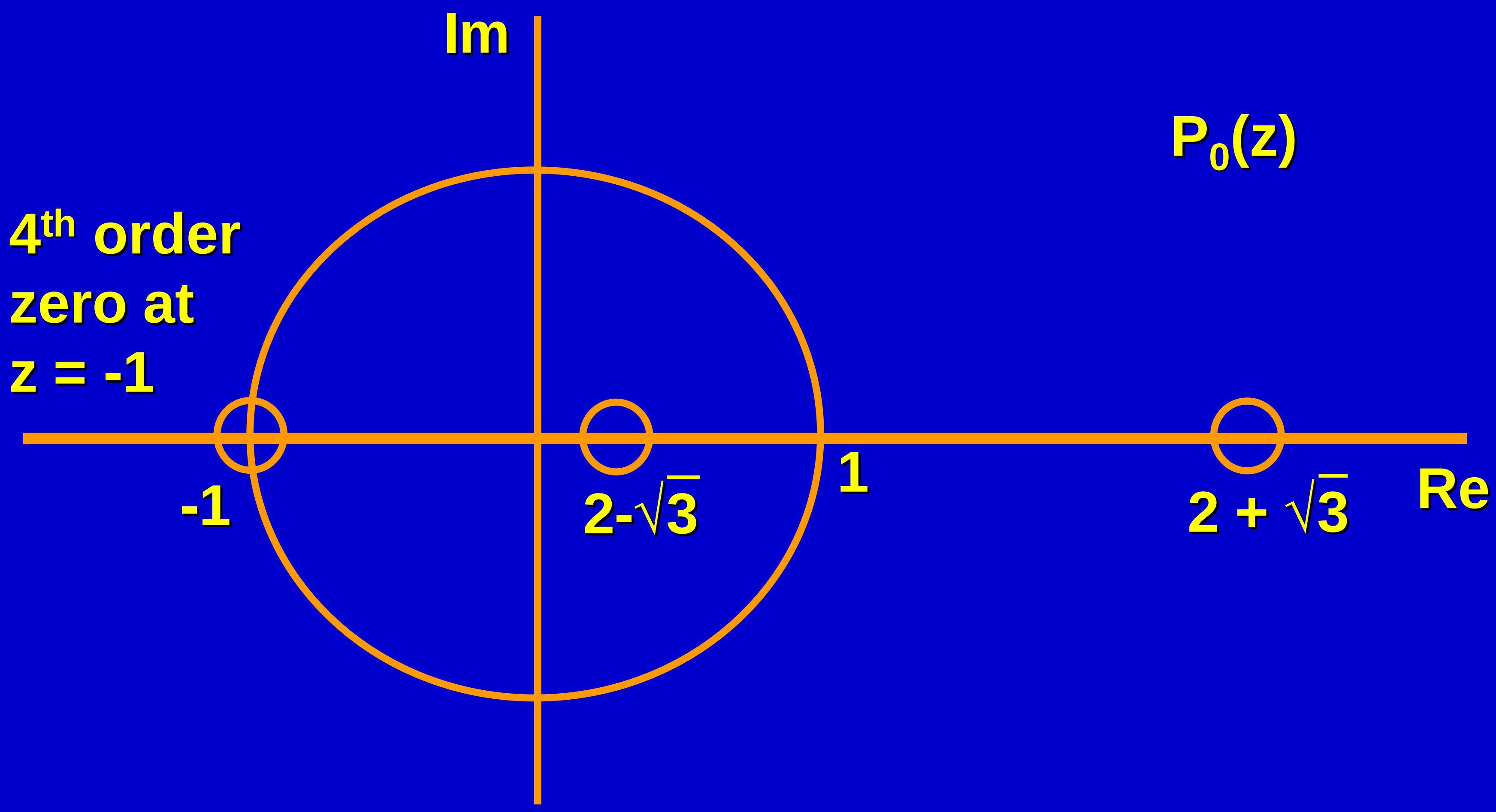
$$= -1/16(1 + z^{-1})^4(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$$

So $P_0(z)$ has zeros at

$z = -1$ (4^{th} order)

$z = 2 \pm \sqrt{3}$

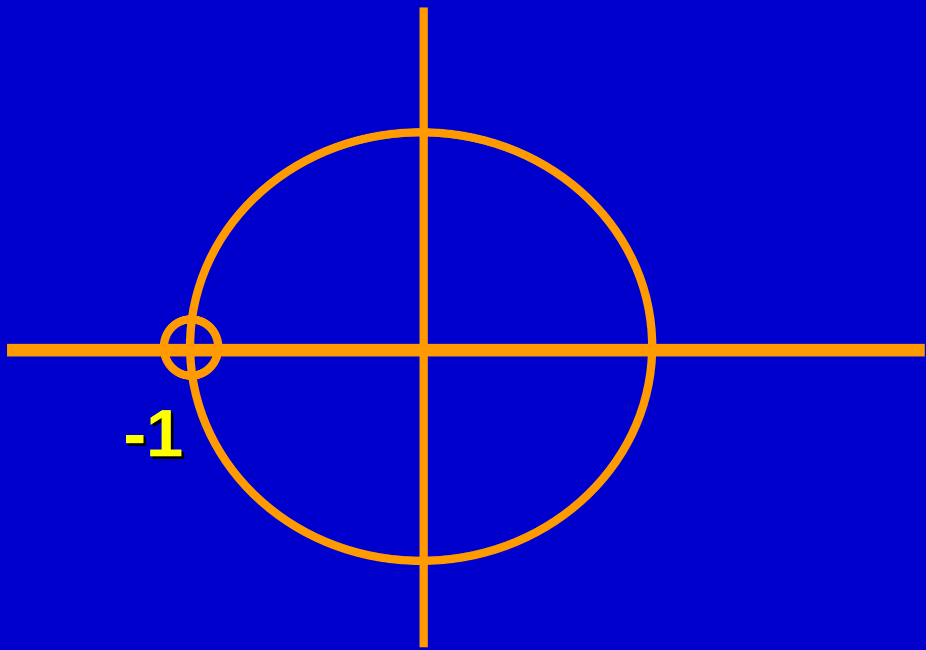
Note: $2 + \sqrt{3} = \frac{1}{2 - \sqrt{3}}$

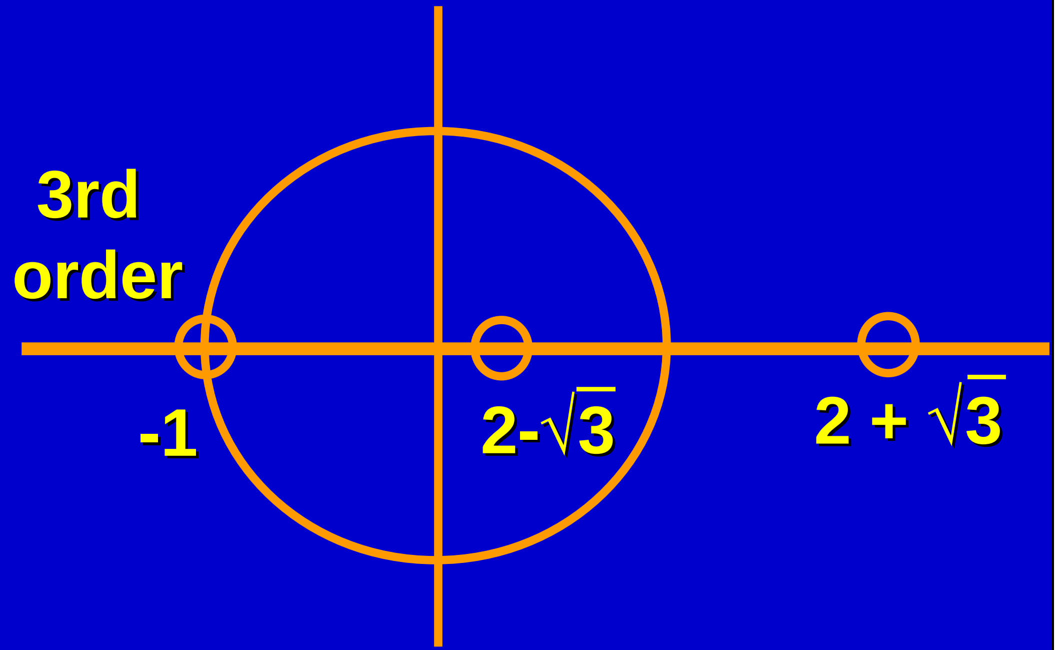


Some possible factorizations

	$H_0(z)$ (or $F_0(z)$)	$F_0(z)$ (or $H_0(z)$)
(a)	1	$-1/16(1 + z^{-1})^4(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(b)	$\frac{1}{2}(1 + z^{-1})$	$-1/8(1 + z^{-1})^3(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(c)	$\frac{1}{4}(1 + z^{-1})^2$	$-1/4(1 + z^{-1})^2(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(d)	$\frac{1}{2}(1 + z^{-1})(2 + \sqrt{3} - z^{-1})$	$-1/8(1 + z^{-1})^3(2 - \sqrt{3} - z^{-1})$
(e)	$1/8(1 + z^{-1})^3$	$-1/2(1 + z^{-1})(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$
(f)	$\frac{(\sqrt{3} - 1)}{4\sqrt{2}}(1 + z^{-1})^2(2 + \sqrt{3} - z^{-1})$	$\frac{-\sqrt{2}}{4(\sqrt{3} - 1)}(1 + z^{-1})^2(2 - \sqrt{3} - z^{-1})$
(g)	$1/16(1 + z^{-1})^4$	$-(2 + \sqrt{3} - z^{-1})(2 - \sqrt{3} - z^{-1})$

Case (b) -- Symmetric filters (linear phase)





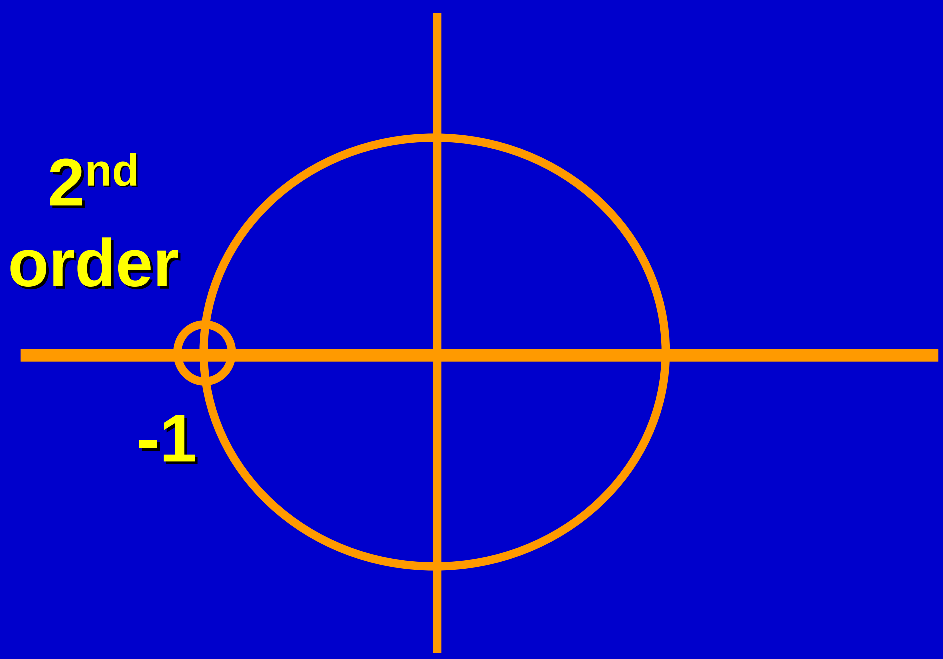
filter length = 2

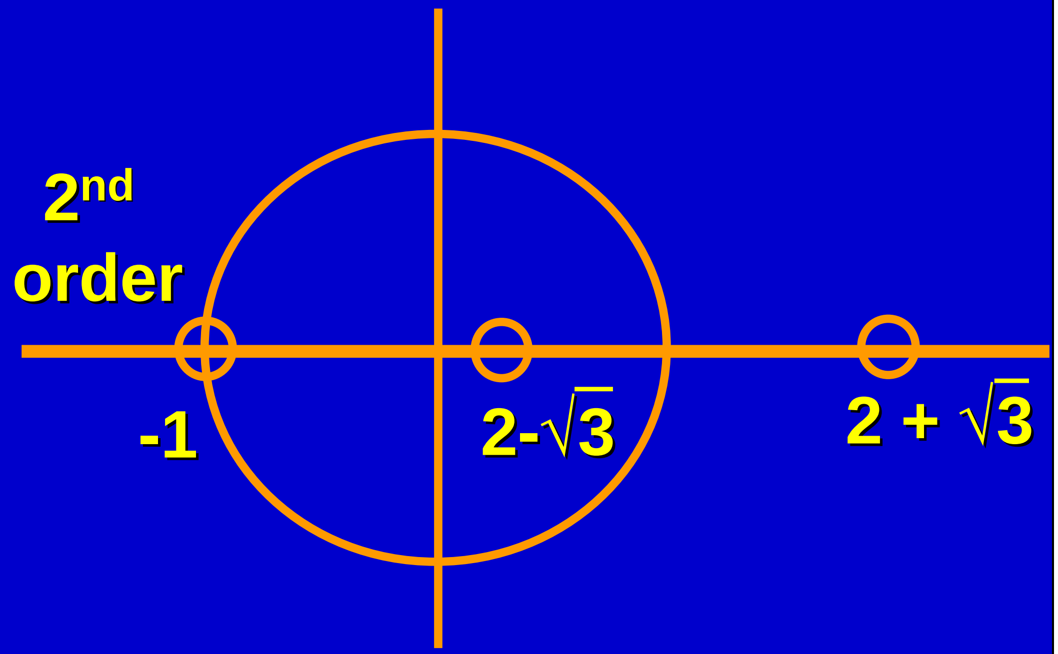
½{ 1, 1 }

filter length = 6

$\frac{1}{8}$ {-1, 1, 8, 8, 1, -1}

Case (c) -- Symmetric filters (linear phase)





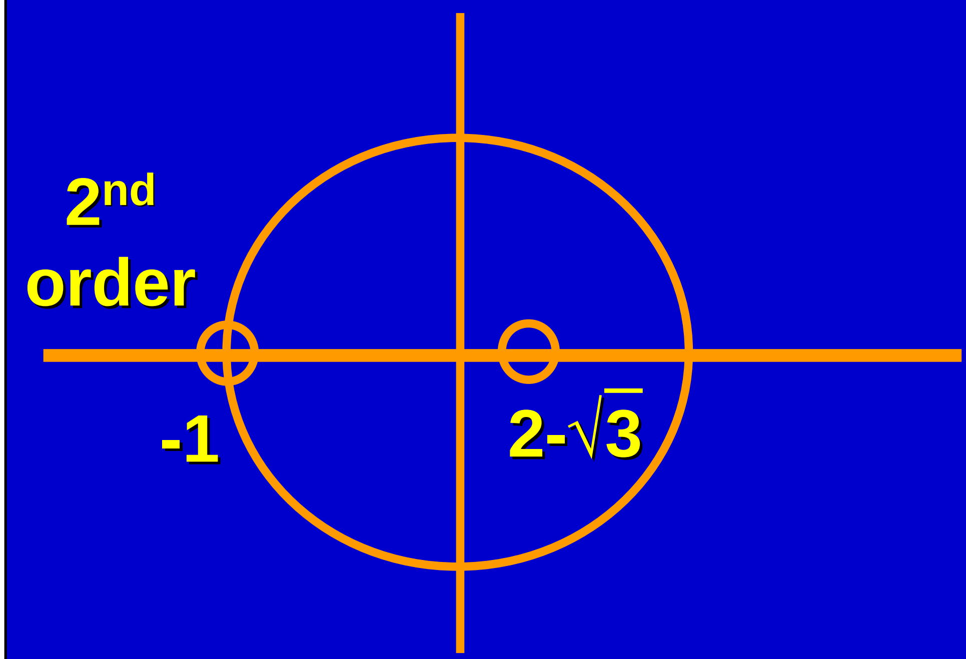
filter length = 3

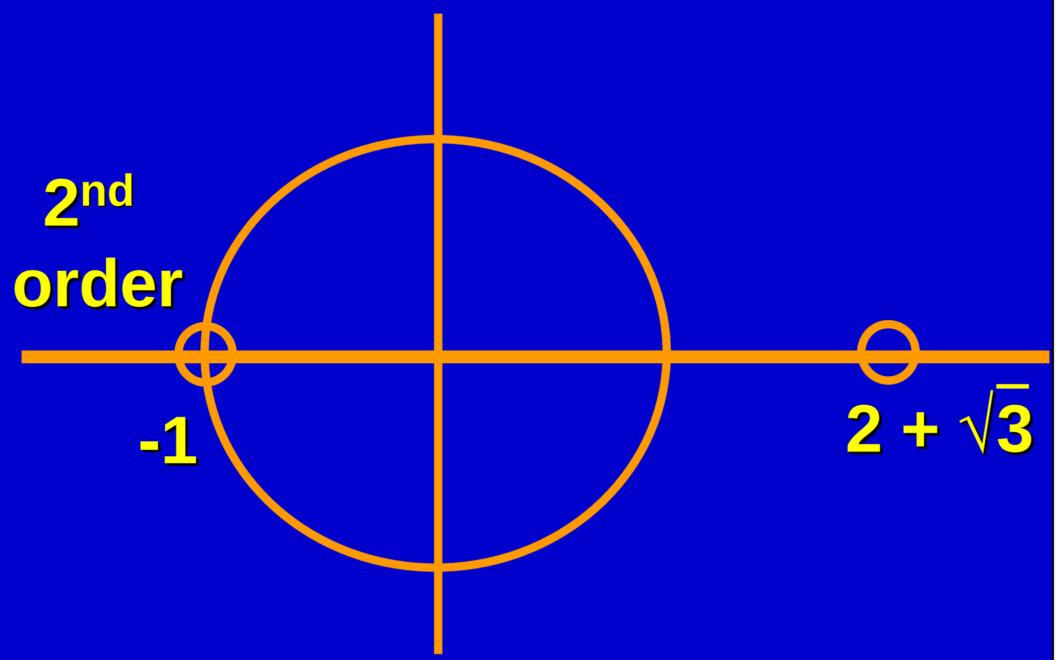
¼ { 1, 2, 1 }

filter length = 5

¼ { -1, 2, 6, 2, -1}

Case (f) -- Orthogonal filters (minimum phase/maximum phase)





filter length = 4

$$\frac{1}{4\sqrt{2}}\left[1+\sqrt{3},\ 3+\sqrt{3},\ 3-\sqrt{3},\ 1-\sqrt{3}\right]$$

filter length = 4

$$\frac{1}{4\sqrt{2}}\left[1-\sqrt{3},\ 3-\sqrt{3},\ 3+\sqrt{3},\ 1+\sqrt{3}\right]$$

Note that, in this case, one filter is the flip (transpose) of the other: $f_0[n] = h_0[3 - n]$

$F_0(z) = z^{-3} H_0(z^{-1})$

General form of product filter (to be derived later):

$$P(z) = 2\left(\frac{1+z}{2}\right)^p \left(\frac{1+z^{-1}}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} \left(\frac{1-z}{2}\right)^k \left(\frac{1-z^{-1}}{2}\right)^k$$

$$P_0(z) = z^{-(2p-1)}\, P(z)$$

$$= (1+z^{-1})^{2p} \frac{1}{2^{2p-1}} \sum_{k=0}^{p-1} \binom{p+k-1}{k} (-1)^k z^{-(p-1)+k} \left(\frac{1-z^{-1}}{2}\right)^{2k}$$

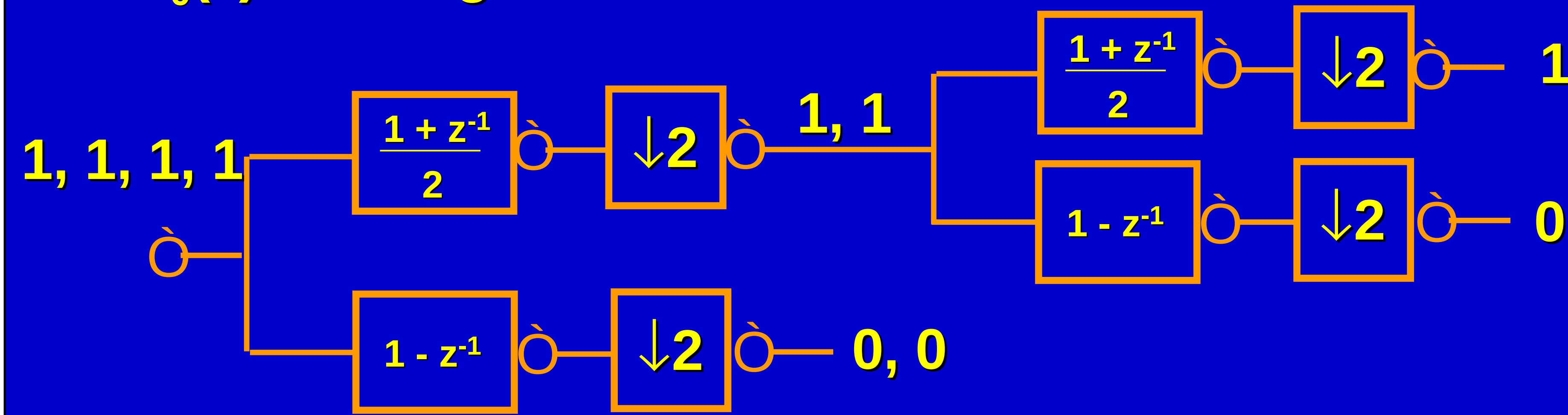
1 2 3 1 4 4 4 4 4 4 2 4 4 4 4 4 4 4 3

Binomial (spline) filter — **Q(z)**: **Cancels all odd powers except $z^{-(2p-1)}$**

$P_0(z)$ has 2p zeros at π (important for stability of iterated filter bank.)
Q(z) factor is needed to ensure perfect reconstruction.

p = 1

$P_0(z)$ has degree 2 $\rightarrow$ leads to Haar filter bank.



$F_0(z) = 1 + z^{-1}$, $H_0(z) = \frac{1 + z^{-1}}{2}$

Synthesis lowpass filter has 1 zero at π

$\rightarrow$ Leads to cancellation of constant signals in analysis highpass channel.

Additional zeros at π would lead to cancellation of higher order polynomials.

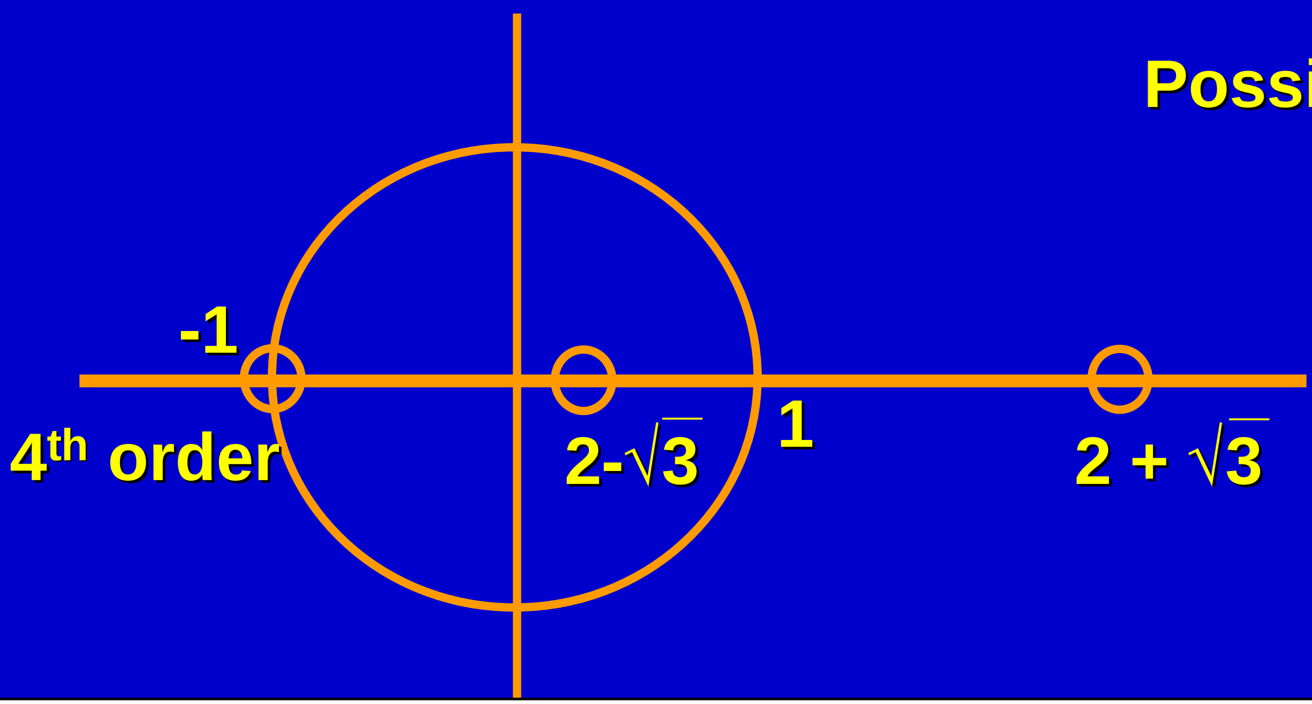
p = 2

$P_0(z)$ has degree $4p - 2 = 6$

$$P_0(z) = (1 + z^{-1})^4 \frac{1}{8} \left\{ \binom{1}{0} z^{-1} - \binom{2}{1}\left(\frac{1 - z^{-1}}{2}\right)^2 \right\}$$

$$= \frac{1}{16}(1 + z^{-1})^4(-1 + 4z^{-1} - z^{-2})$$

$$= \frac{1}{16}\{-1 + 9z^{-2} + 16z^{-3} + 9z^{-4} - z^{-6}\}$$



Possible factorizations

1/8 trivial

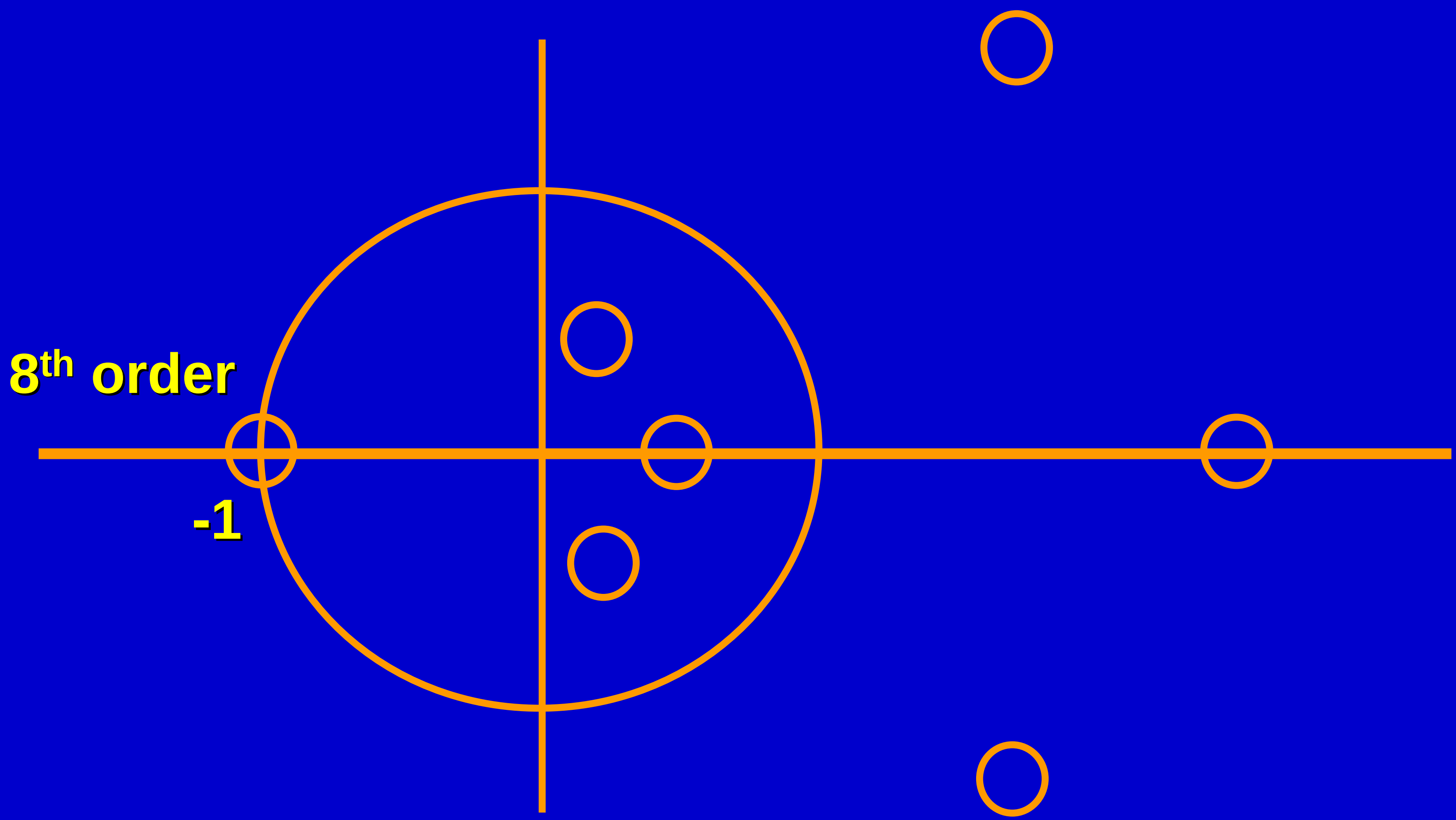
2/6 } linear phase

3/5

4/4 orthogonal (Daubechies-4)

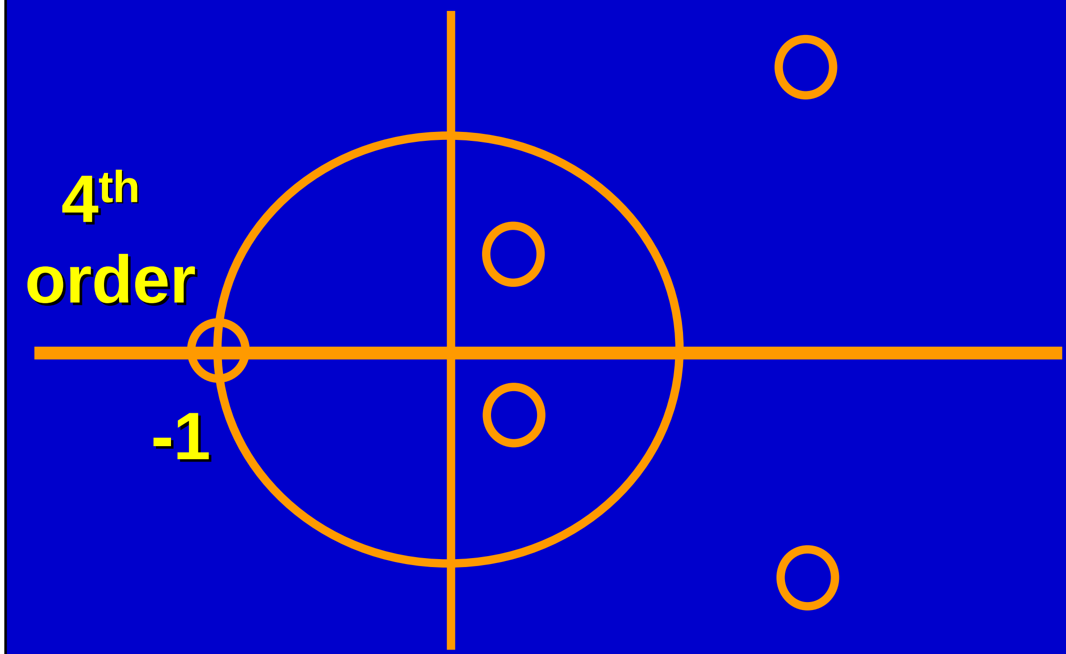
$p = 4$

$P_0(z)$ has degree $4p - 2 = 14$

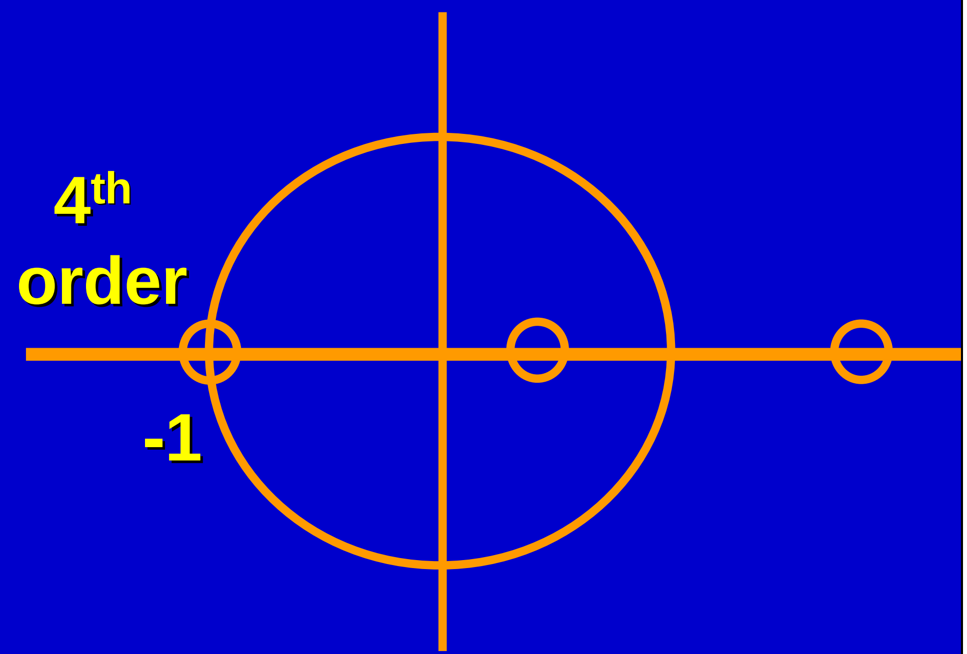


Common factorizations (p = 4):

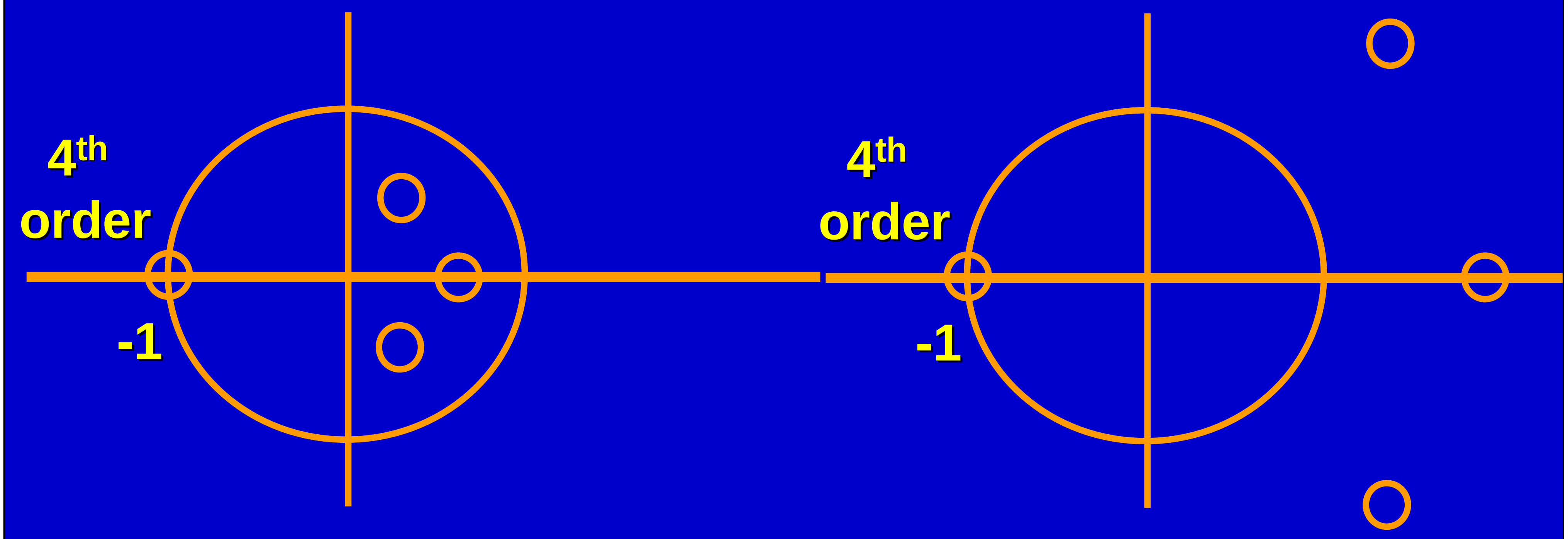
(a) 9/7



Known in Matlab as bior4.4



(b) 8/8 (Daubechies 8) -- Known in Matlab as db4

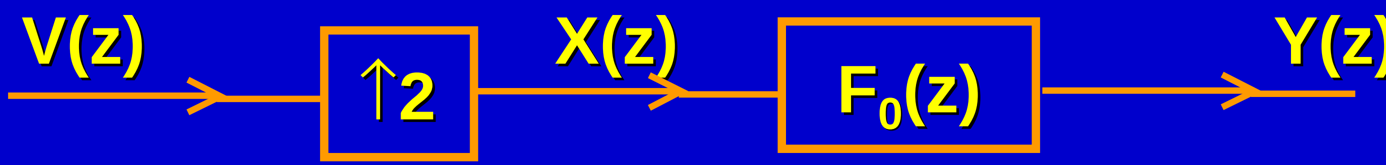


Why choose a particular factorization?

Consider the example with p = 2:

i. **One of the factors is halfband**

The trivial 1/8 factorization is generally not desirable, since each factor should have at least one zero at π. However, the fact that $F_0(z)$ is halfband is interesting in itself.



Let $F_0(z)$ be centered, for convenience. Then

$$F_0(z) = 1 + \text{odd powers of } z$$

Now

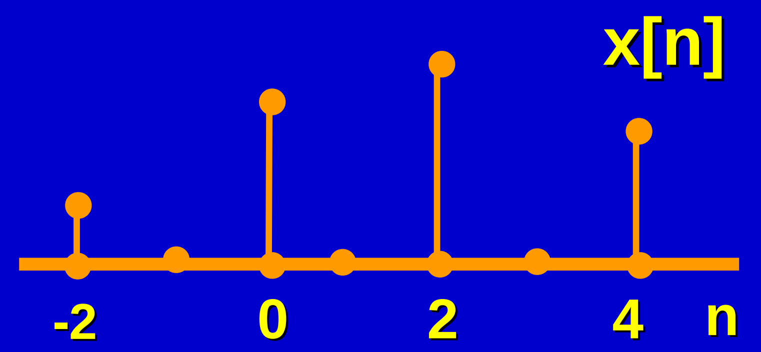
$$X(z) = V(z^2) = \text{even powers of } z \text{ only}$$

So

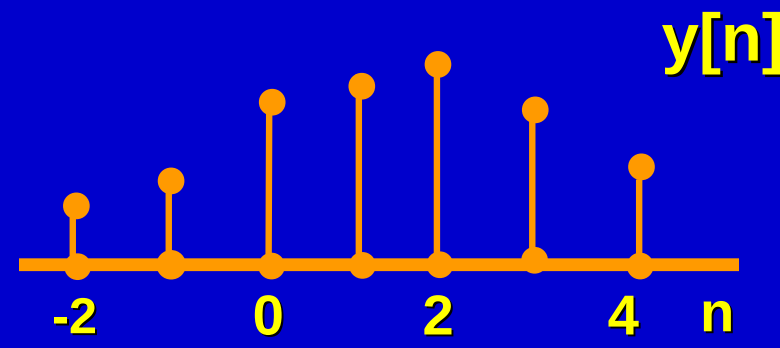
$$Y(z) = F_0(z)\,X(z)$$
$$= X(z) + \text{odd powers}$$

$$y[n] = \begin{cases} x[n] & ; \quad n \text{ even} \\ \sum_{k \text{ odd}} f_0[k]\,x[n-k] & ; \quad n \text{ odd} \end{cases}$$

$\Rightarrow$ $f_0[n]$ is an interpolating filter



Another example: $f_0[n] = \dfrac{\sin(\frac{\pi}{2})n}{\pi n}$
(ideal bandlimited interpolating filter)



ii. **Linear phase factorization e.g. 2/6, 5/3**

Symmetric (or antisymmetric) filters are desirable for many applications, such as image processing. All frequencies in the signal are delayed by the same amount i.e. there is no phase distortion.

$$h[n] \text{ linear phase} \Rightarrow A(\omega)e^{-i(\omega\alpha + \theta)}$$

- $A(\omega)$: real
- α: delays all frequencies by α samples
- θ: 0 if symmetric; $\frac{\pi}{2}$ if antisymmetric

Linear phase may not necessarily be the best choice for audio applications due to preringing effects.

iii. **Orthogonal factorization**
This leads to a minimum phase filter and a maximum phase filter, which may be a better choice for applications such as audio. The orthogonal factorization leads to the Daubechies family of wavelets – a particularly neat and interesting case. 4/4 factorization:

$$H_0(z) = \frac{\sqrt{3}-1}{4\sqrt{2}}(1+z^{-1})^2[(2+\sqrt{3})-z^{-1}]$$

$$= \frac{1}{4\sqrt{2}}\{(1+\sqrt{3})+(3+\sqrt{3})z^{-1}+(3-\sqrt{3})z^{-2}+(1-\sqrt{3})z^{-3}\}$$

$$F_0(z) = \frac{-\sqrt{2}}{4(\sqrt{3}-1)}(1+z^{-1})^2[(2-\sqrt{3})-z^{-1}]$$

$$= \frac{\sqrt{3}-1}{4\sqrt{2}}z^{-3}(1+z^2)[(2+\sqrt{3})-z]$$

$$= z^{-3}H_0(z^{-1})$$

$$P(z) = z^{l} P_0(z)$$

$$= H_0(z)\, H_0(z^{-1})$$

From alias cancellation condition:

$$H_1(z) = F_0(-z) = -z^{-3} H_0(-z^{-1})$$

$$F_1(z) = -H_0(-z) = z^{-3} H_1(z^{-1})$$

Special Case: Orthogonal Filter Banks

Choose $H_1(z)$ so that

$$H_1(z) = - z^{-N} H_0(- z^{-1})$$ N odd

Time domain

$$h_1[n] = (- 1)^n h_0[N - n]$$

$F_0(z) = H_1(- z) = z^{-N} H_0(z^{-1})$

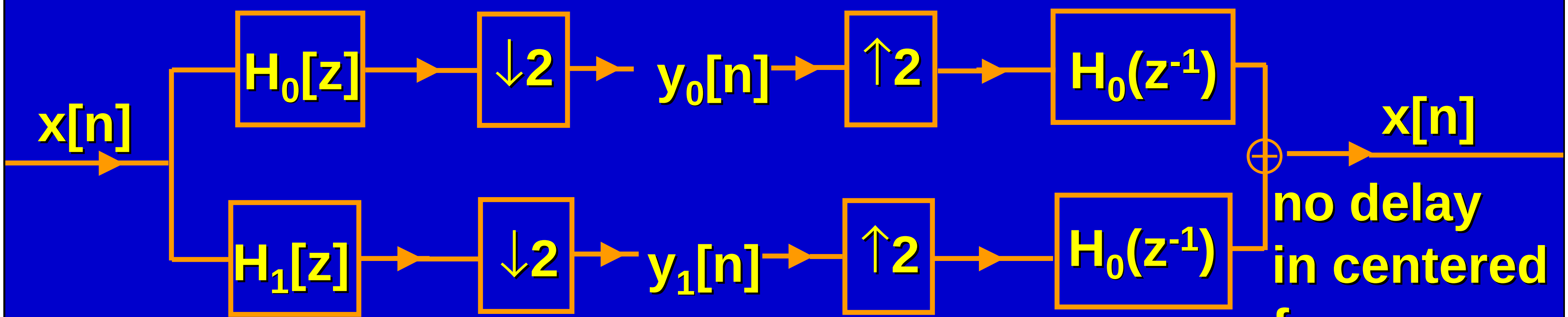
$\Rightarrow \quad f_0[n] = h_0[N - n]$

$F_1(z) = - H_0(- z) = z^{-N} H_1(z^{-1})$

$\Rightarrow f_1[n] = h_1[N - n]$

So the synthesis filters, $f_k[n]$, are just the time-reversed versions of the analysis filters, $h_k[n]$, with a delay.

Why is the Daubechies factorization orthogonal?
Consider the centered form of the filter bank:



Analysis bank
causal – only
negative powers
of z

Synthesis bank
anticausal – only
positive powers
of z

In matrix form:

Analysis

$$\begin{bmatrix} y_o \\ \hline y_1 \end{bmatrix} = \underbrace{\begin{bmatrix} L \\ \hline B \end{bmatrix}}_{W} \begin{bmatrix} x \end{bmatrix}$$

1 2 3

Synthesis

$$\begin{bmatrix} x \end{bmatrix} = \underbrace{\begin{bmatrix} L^T & \vert & B^T \end{bmatrix}}_{W^T} \begin{bmatrix} y_0 \\ \hline y_1 \end{bmatrix}$$

1 2 3

So

$$x = W^T W x \quad \text{for any } x$$

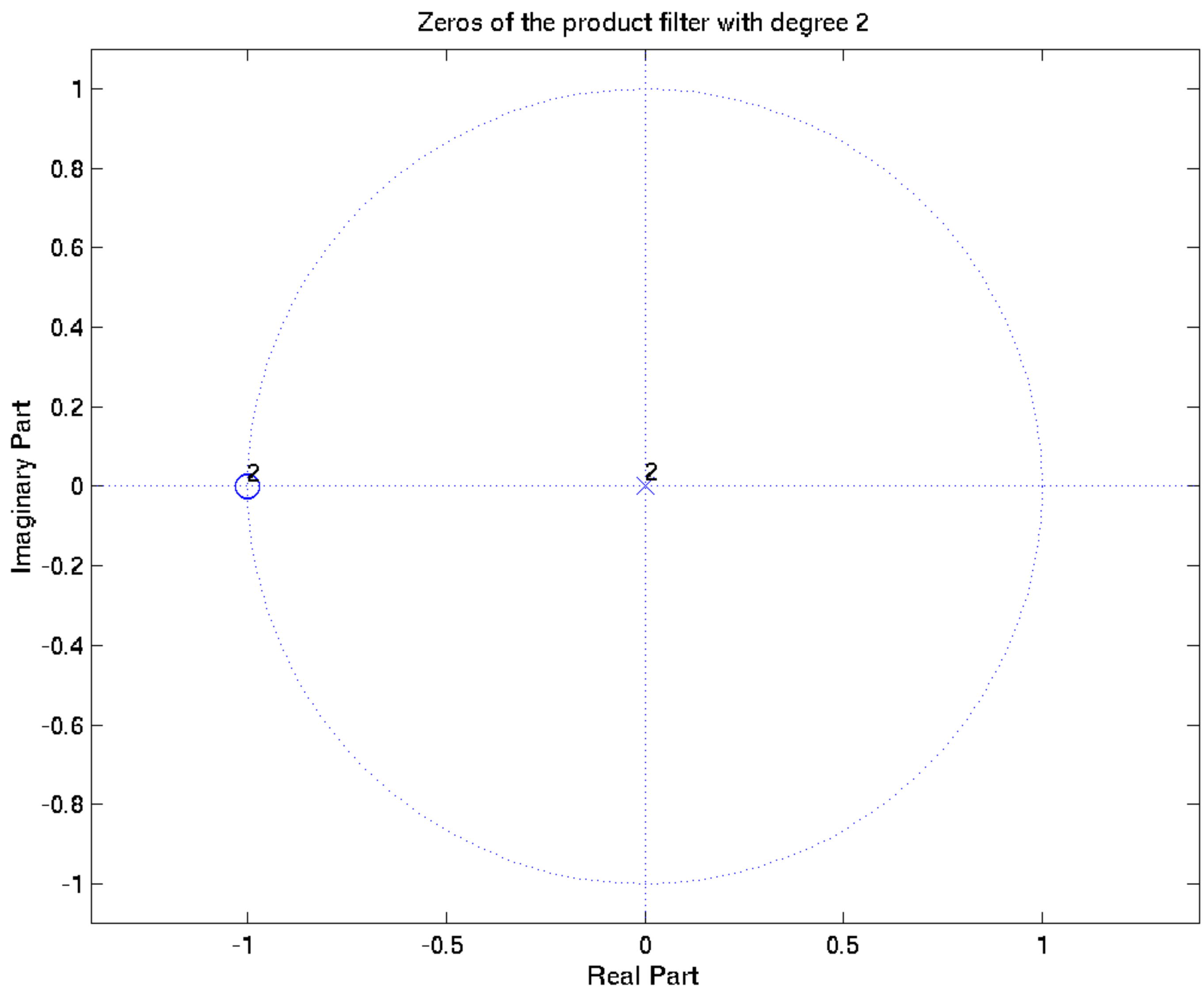
$$W^T W = I = W W^T$$

An important fact: symmetry prevents orthogonality

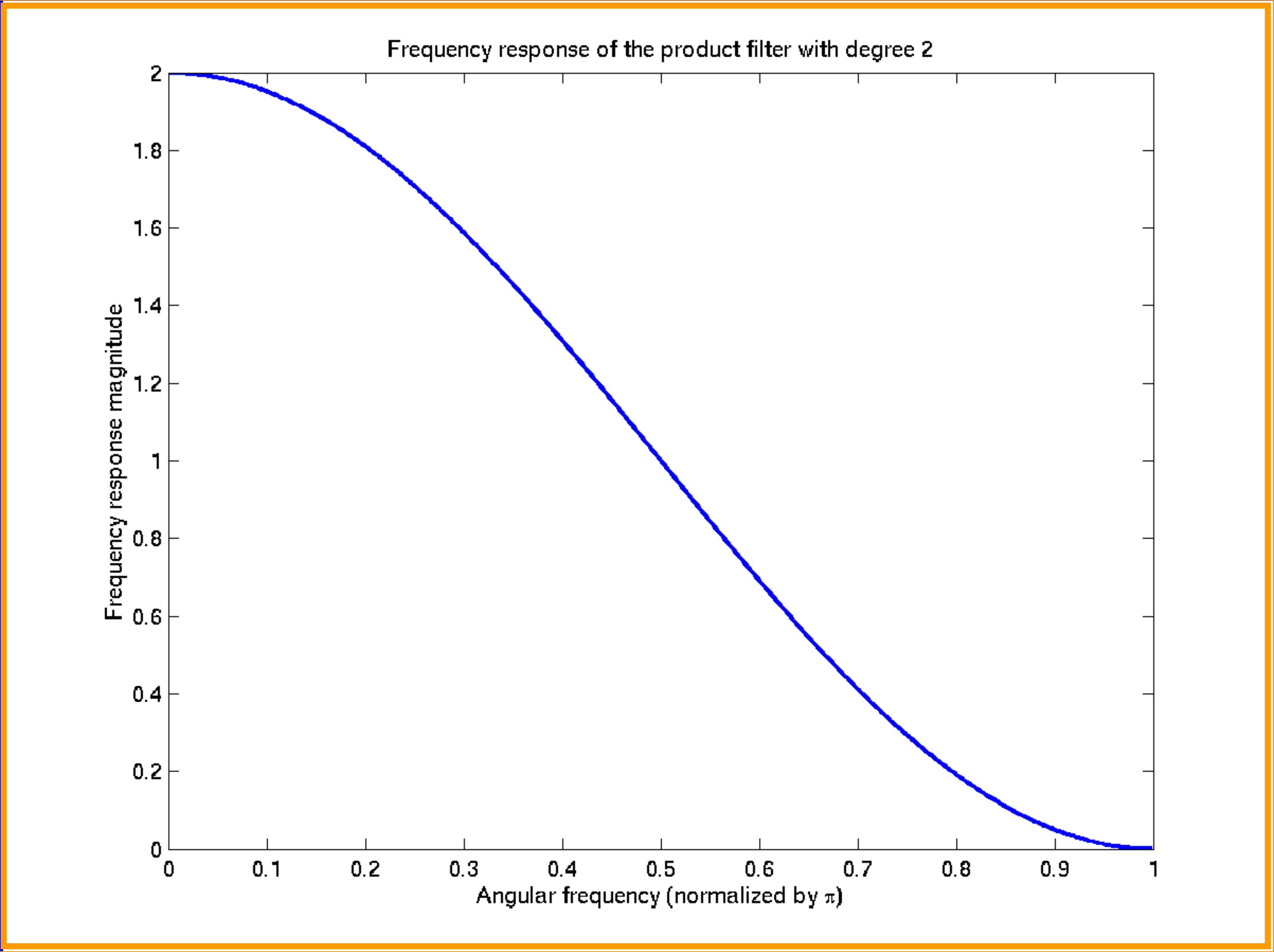
Matlab Example 2

1. Product filter examples

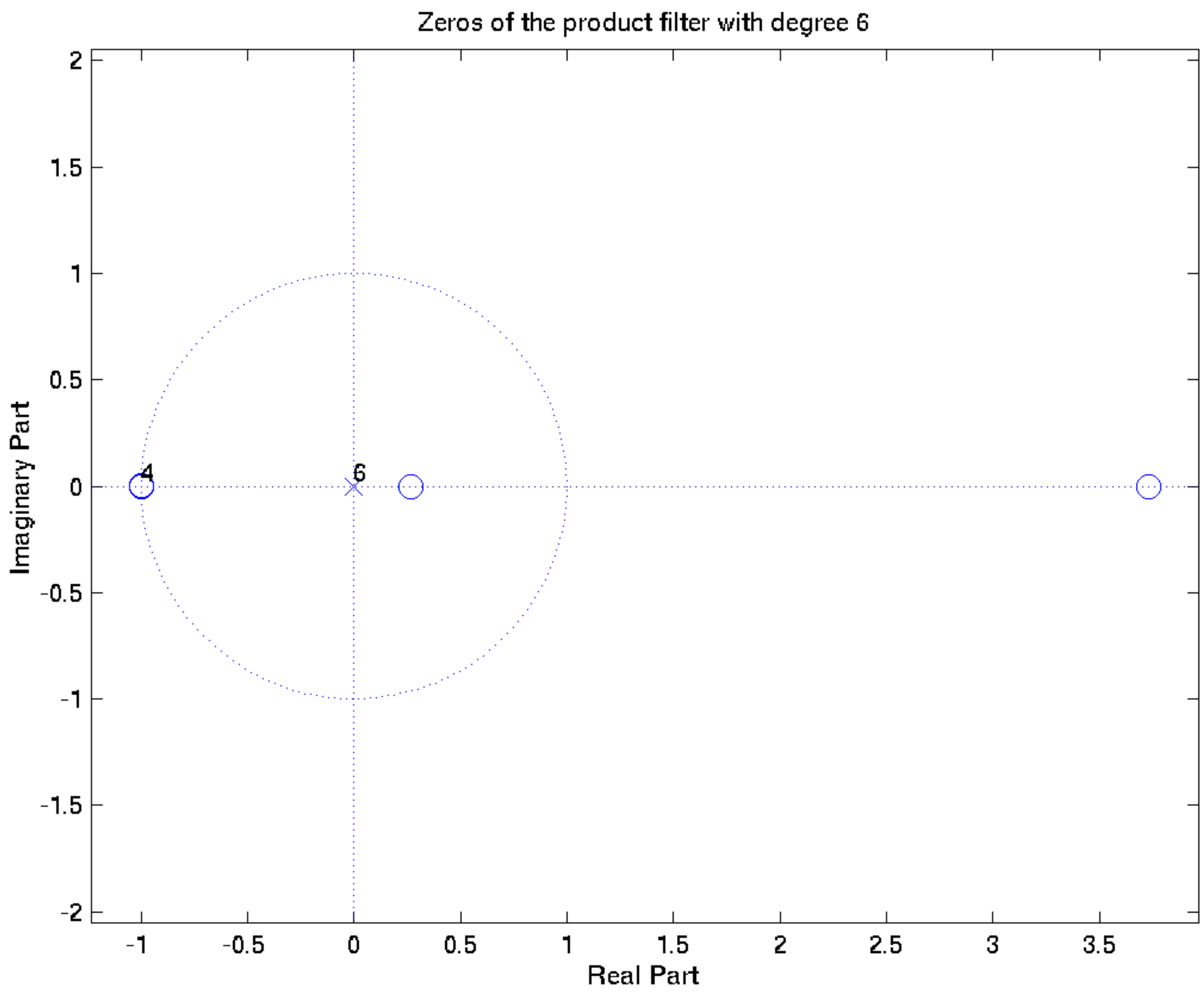
Degree-2 (p=1): pole-zero plot



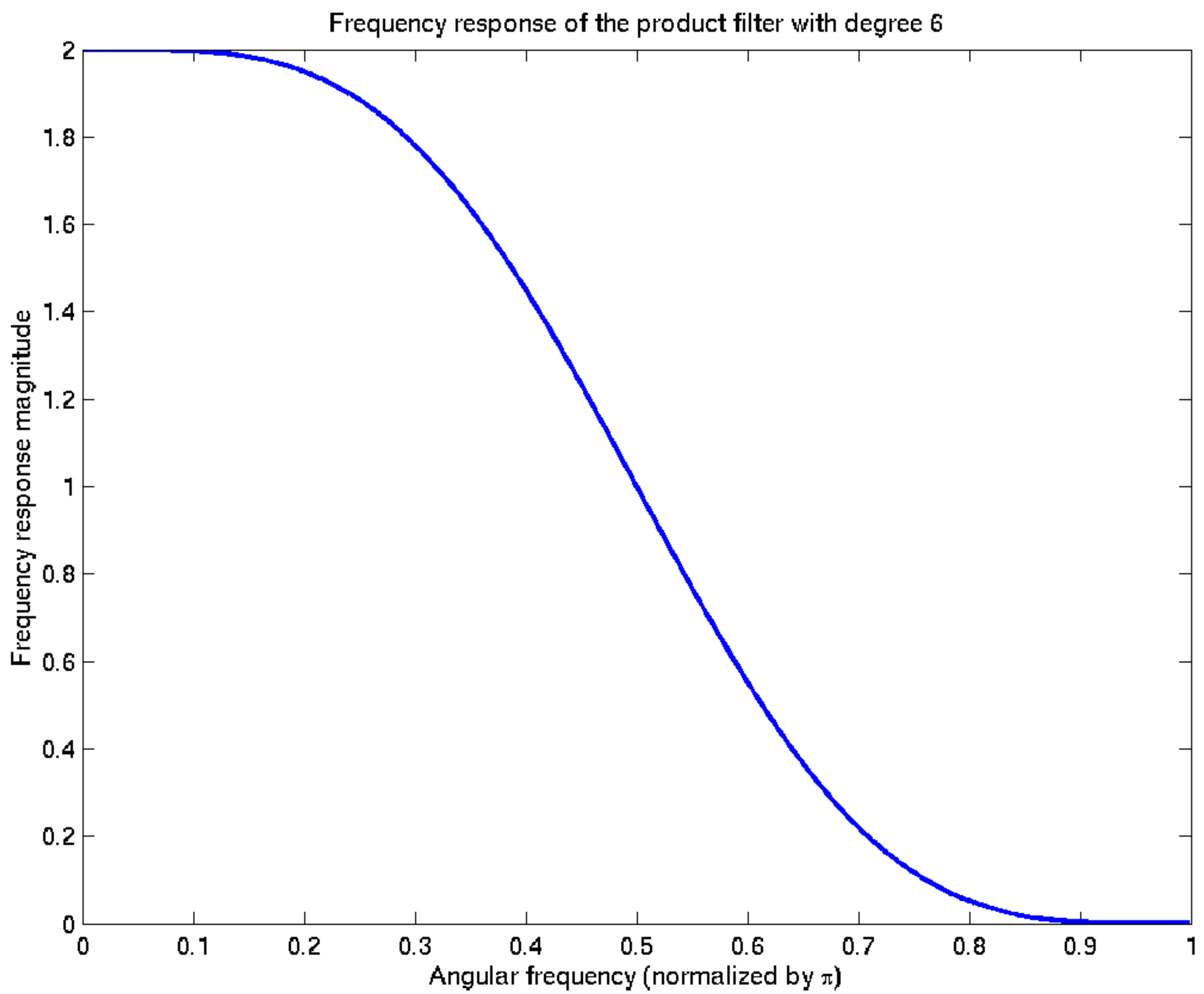
Degree-2 (p=1): Freq. response



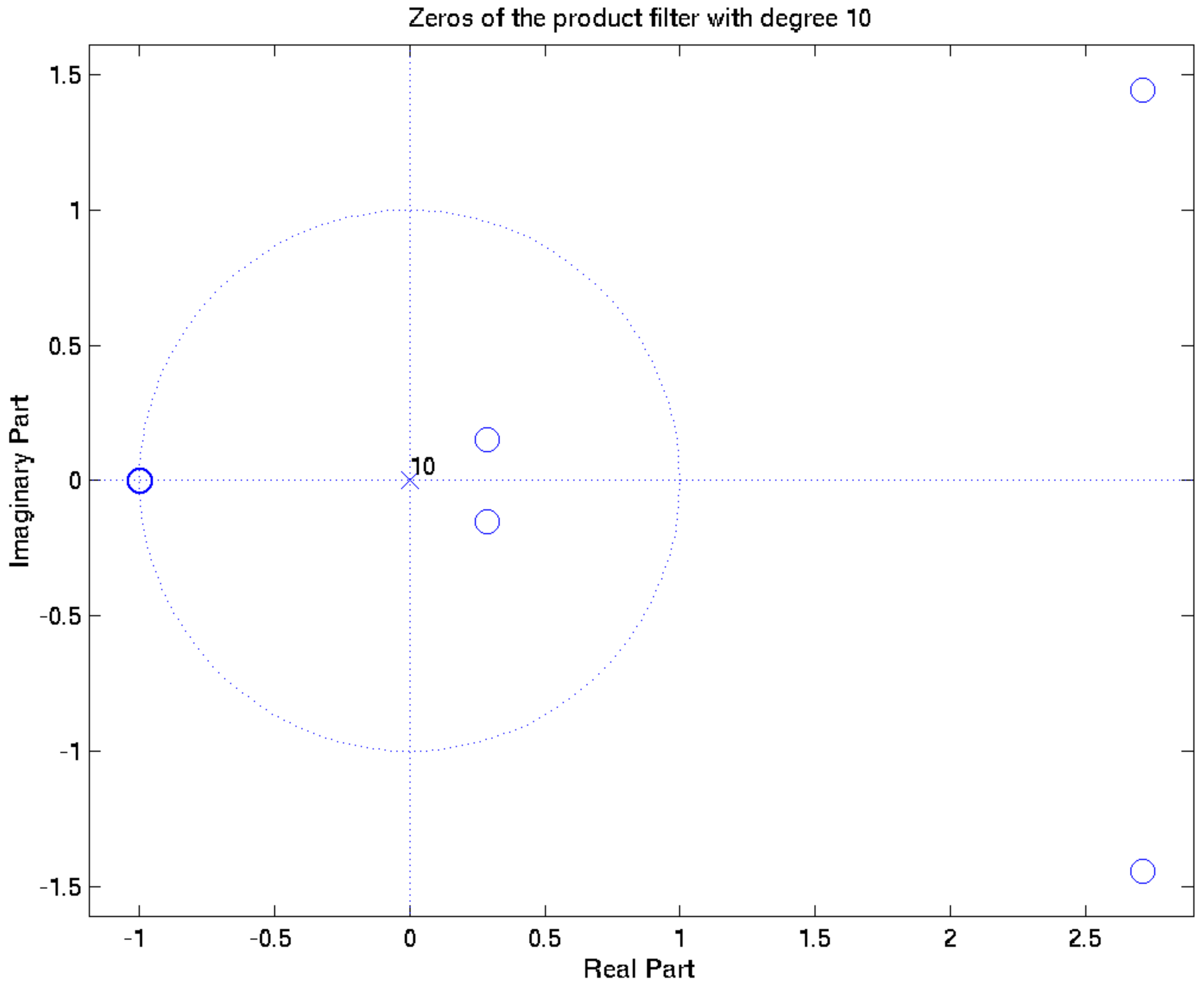
Degree-6 (p=2): pole-zero plot



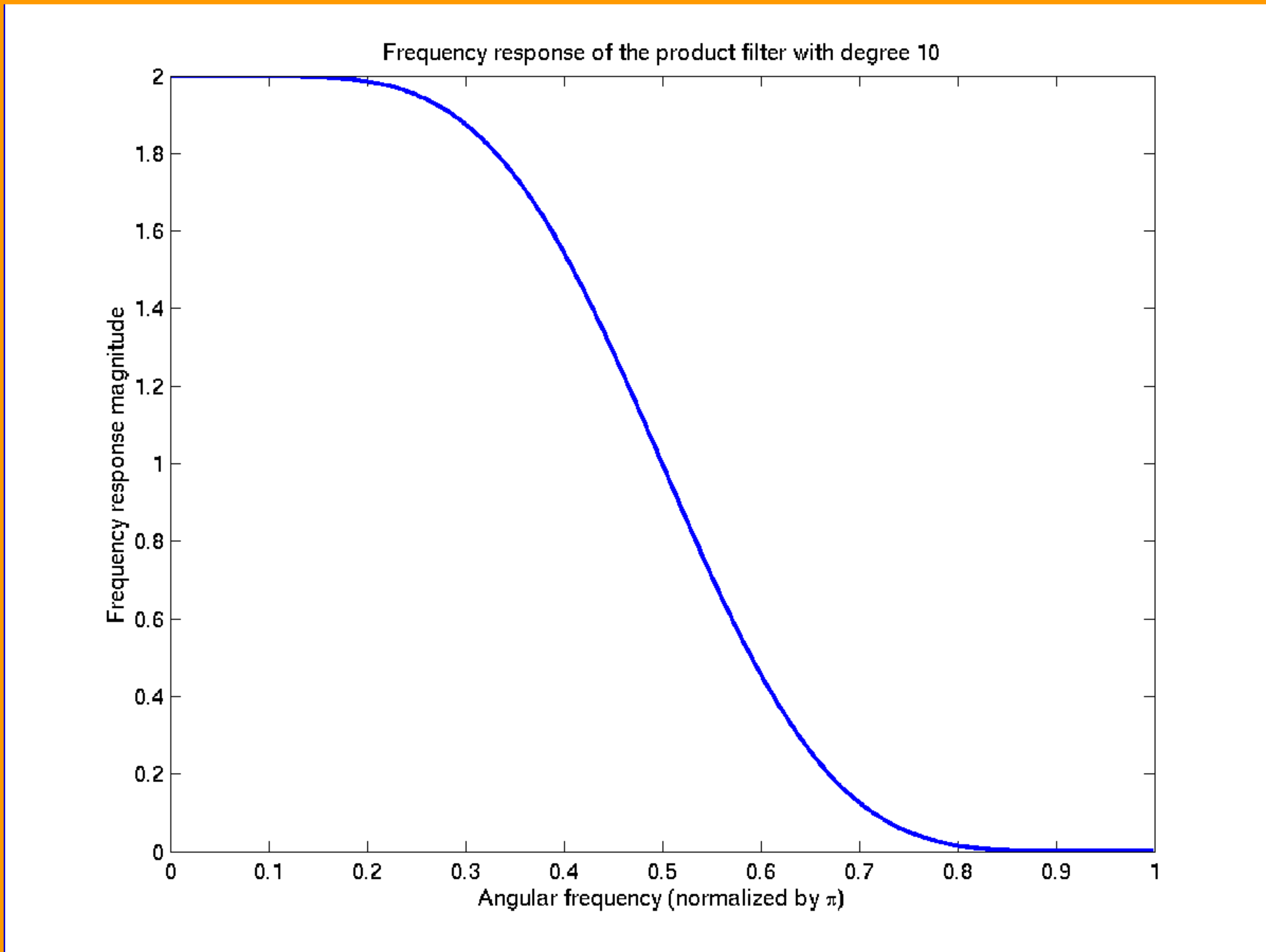
Degree-6 (p=2): Freq. response



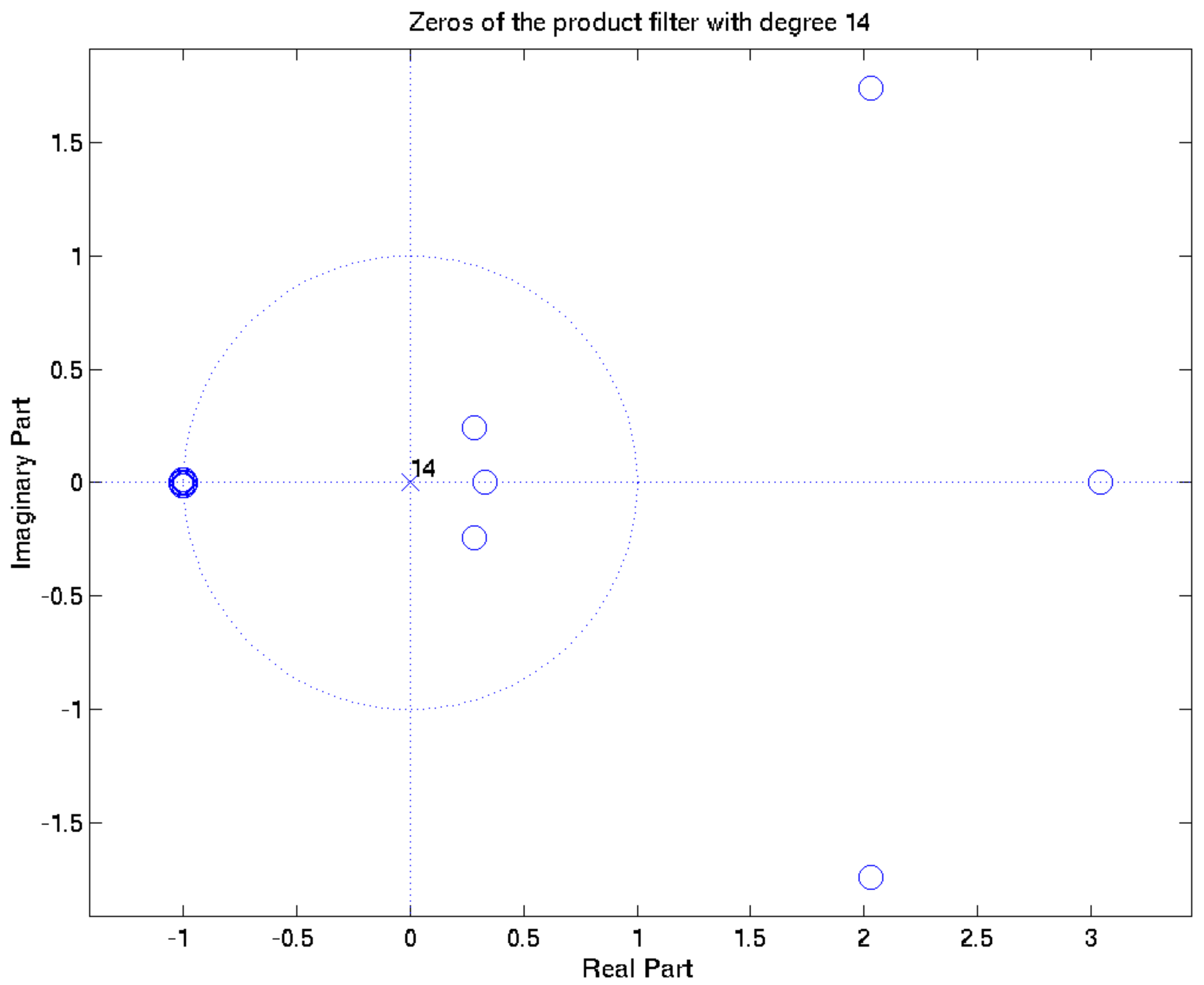
Degree-10 (p=3): pole-zero plot



Degree-10 (p=3): Freq. response



Degree-14 (p=4): pole-zero plot



Degree-14 (p=4): Freq. response