Course 18.327 and 1.130
Wavelets and Filter Banks

Orthogonal Filter Banks;
Paraunitary Matrices;
Orthogonality Condition (Condition O) in the Time Domain, Modulation Domain and Polyphase Domain

Unitary Matrices

The constant complex matrix A is said to be unitary if

$$A^{\dagger} A = I$$

example:

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix} \qquad A^* = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$$

$$A^{-1} = \frac{-1}{\sqrt{2}} \begin{bmatrix} -1 & i \\ -i & 1 \end{bmatrix} \qquad A^{\dagger} = A^{*T} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ i & -1 \end{bmatrix}$$

$$\Rightarrow A^{\dagger} = A^{-1}$$

Paraunitary Matrices

The matrix function H(z) is said to be paraunitary if it is unitary for all values of the parameter z

$$H^T(z^{-1})\ H(z) = I \qquad \text{for all } z \neq 0 \text{ -----------------(1)}$$

Frequency Domain:

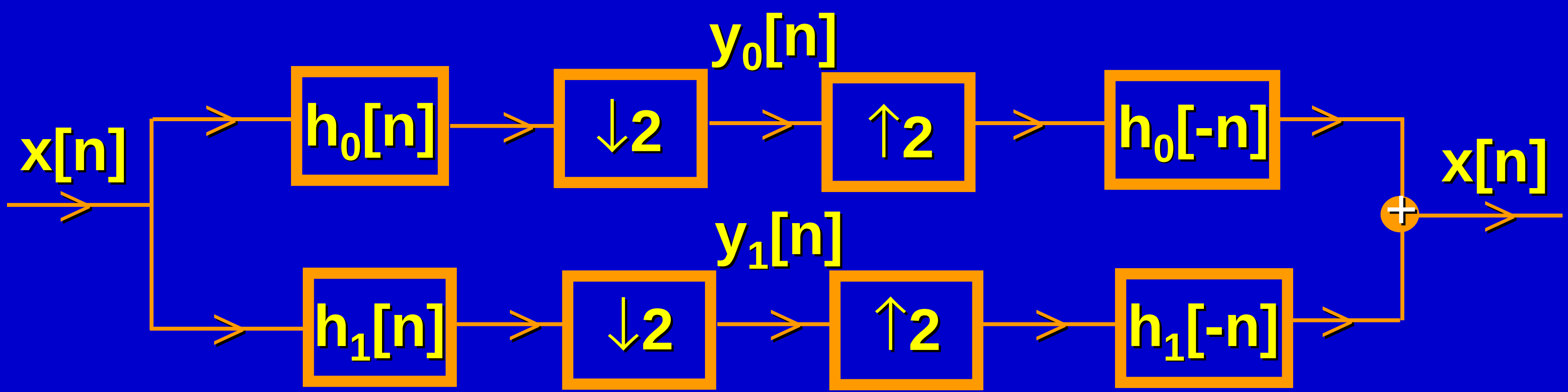
$$H^T(-\omega)\ H(\omega) = I \qquad \text{for all } \omega$$

or $H^{*T}(\omega)\ H(\omega) = I$

Note: we are assuming that h[n] are real.

Orthogonal Filter Banks

Centered form (PR with no delay):



Synthesis bank = transpose of analysis bank

$h_0[n]$ causal $\Rightarrow f_0[n] \equiv h_0[-n]$ anticausal

What are the conditions on $h_0[n]$, $h_1[n]$, in the

(i) time domain?
(ii) polyphase domain?
(iii) modulation domain?

Time Domain

Analysis: N = 3 (filter length = 4)

$$
\begin{bmatrix} M \\ y_0[0] \\ y_0[1] \\ y_0[2] \\ y_0[3] \\ M \\ \hline M \\ y_1[0] \\ y_1[1] \\ y_1[2] \\ y_1[3] \\ M \end{bmatrix}
=
\begin{bmatrix}
K & & & & & & & & & \\
h_0[3] & h_0[2] & h_0[1] & h_0[0] & & & & & & \\
& & h_0[3] & h_0[2] & h_0[1] & h_0[0] & & & & \\
& & & & h_0[3] & h_0[2] & h_0[1] & h_0[0] & & \\
& & & & & & h_0[3] & h_0[2] & h_0[1] & h_0[0] \\
& & & & & & & & & L \\
\hline
K & & & & & & & & & \\
h_1[3] & h_1[2] & h_1[1] & h_1[0] & & & & & & \\
& & h_1[3] & h_1[2] & h_1[1] & h_1[0] & & & & \\
& & & & h_1[3] & h_1[2] & h_1[1] & h_1[0] & & \\
& & & & & & h_1[3] & h_1[2] & h_1[1] & h_1[0] \\
& & & & & & & & & L
\end{bmatrix}
\begin{bmatrix} M \\ x[-3] \\ x[-2] \\ x[-1] \\ x[-0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ M \end{bmatrix}
$$

1 2 3

W

-----(2)

Synthesis:

$$
\begin{bmatrix} \vdots \\ x[-3] \\ x[-2] \\ x[-1] \\ x[0] \\ x[1] \\ x[2] \\ x[3] \\ x[4] \\ x[5] \\ x[6] \\ \vdots \end{bmatrix}
=
\left[\begin{array}{cccc|cccc}
\vdots & & & & \vdots & & & \\
h_0[3] & & & & h_1[3] & & & \\
h_0[2] & & & & h_1[2] & & & \\
h_0[1] & h_0[3] & & & h_1[1] & h_1[3] & & \\
h_0[0] & h_0[2] & & & h_1[0] & h_1[2] & & \\
 & h_0[1] & h_0[3] & & & h_1[1] & h_1[3] & \\
 & h_0[0] & h_0[2] & & & h_1[0] & h_1[2] & \\
 & & h_0[1] & h_0[3] & & & h_1[1] & h_1[3] \\
 & & h_0[0] & h_0[2] & & & h_1[0] & h_1[2] \\
 & & & h_0[1] & & & & h_1[1] \\
 & & & h_0[0] \ \vdots & & & & h_1[0] \ \vdots
\end{array}\right]
\begin{bmatrix} \vdots \\ y_0[0] \\ y_0[1] \\ y_0[2] \\ y_0[3] \\ \vdots \\ \hline \vdots \\ y_1[0] \\ y_1[1] \\ y_1[2] \\ y_1[3] \\ \vdots \end{bmatrix}
$$

1 2 3

-----(3)

W^T

Orthogonality condition (Condition O) is

$$W^T W = I = W W^T \Rightarrow W \text{ orthogonal matrix}$$

Block Form:

$$W = \begin{bmatrix} L \\ B \end{bmatrix}$$

$$L^T L + B^T B = I$$

$$\begin{bmatrix} LL^T & LB^T \\ BL^T & BB^T \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

$$LL^T = I \Rightarrow \sum_n h_0[n]\, h_0[n-2k] = \delta[k] \quad \text{-------------(4)}$$

$$LB^T = 0 \Rightarrow \sum_n h_0[n]\, h_1[n-2k] = 0 \quad \text{-------------(5)}$$

$$BB^T = I \Rightarrow \sum_n h_1[n]\, h_1[n-2k] = \delta[k] \quad \text{-------------(6)}$$

Good choice for $h_1[n]$:

$$h_1[n] = (-1)^n h_0[N-n] \qquad ; \; N \text{ odd} \quad \text{---------------}(7)$$

→ **Alternating flip**

Example: N = 3

$$h_1[0] = h_0[3]$$
$$h_1[1] = -h_0[2]$$
$$h_1[2] = h_0[1]$$
$$h_1[3] = -h_0[0]$$

With this choice, Equation (5) is automatically satisfied:

$$k = -1: h_0[0]h_0[1] - h_0[1]h_0[0] = 0$$
$$k = 0: h_0[0]h_0[3] - h_0[1]h_0[2] + h_0[2]h_0[1] - h_0[3]h_0[0] = 0$$
$$k = 1: h_0[2]h_0[3] - h_0[3]h_0[2] = 0$$

$k = \pm 2$: no overlap

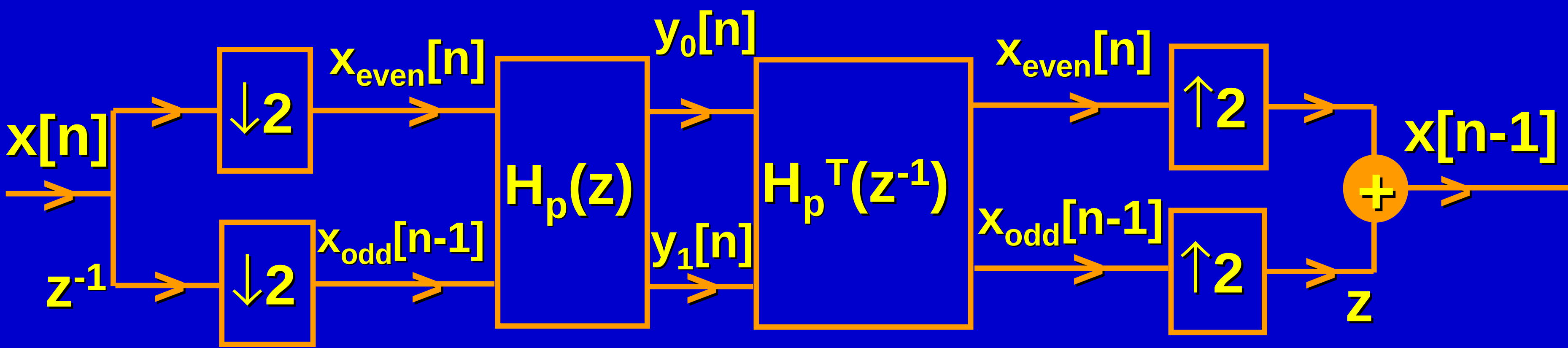
Also, Equation (6) reduces to Equation (4)

$$\delta[k] = \sum_n h_1[n]\, h_1[n-2k] = \sum_n (-1)^n\, h_0[N-n]\, (-1)^{n-2k}\, h_0[N-n+2k]$$

$$= \sum_l h_0[l]\, h_0[l + 2k]$$

So, Condition O on the lowpass filter + alternating flip for highpass filter lead to orthogonality

Polyphase Domain



$$H_p(z) = \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix} \longrightarrow$$

Polyphase Matrix

Condition O:

$$H_p^T(z^{-1})\ H_p(z) = I \Rightarrow H_p(z) \text{ is paraunitary}$$

$$\begin{bmatrix} H_{0,even}(z^{-1}) & H_{1,even}(z^{-1}) \\ H_{0,odd}(z^{-1}) & H_{1,odd}(z^{-1}) \end{bmatrix} \begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Reverse the order of multiplication:

$$\begin{bmatrix} H_{0,even}(z) & H_{0,odd}(z) \\ H_{1,even}(z) & H_{1,odd}(z) \end{bmatrix} \begin{bmatrix} H_{0,even}(z^{-1}) & H_{1,even}(z^{-1}) \\ H_{0,odd}(z^{-1}) & H_{1,odd}(z^{-1}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Express Condition O as a condition on $H_{0,even}(z)$, $H_{0,odd}(z)$:

$$H_{0,even}(z)\ H_{0,even}(z^{-1}) + H_{0,odd}(z)\ H_{0,odd}(z^{-1}) = 1 \qquad \text{------(8)}$$

Frequency domain:

$$|H_{0,even}(\omega)|^2 + |H_{0,odd}(\omega)|^2 = 1 \qquad \text{-------(9)}$$

The alternating flip construction for $H_1(z)$ ensures that the remaining conditions are satisfied.

$$H_0(z) = H_{0,even}(z^2) \;+\; z^{-1}H_{0,odd}(z^2)$$

$$H_1(z) = -z^{-N}\, H_0(-z^{-1}) \qquad\qquad \textbf{alternating flip}$$

$$= -z^{-N}\,\{H_{0,even}(z^{-2}) \;-\; z\, H_{0,odd}(z^{-2})\}$$

$$= \underbrace{-z^{-N}\, H_{0,even}(z^{-2})}_{z^{-1}\, H_{1,odd}(z^2)} \;+\; \underbrace{z^{-N+1}\, H_{0,odd}(z^{-2})}_{H_{1,even}(z^2)}$$

So

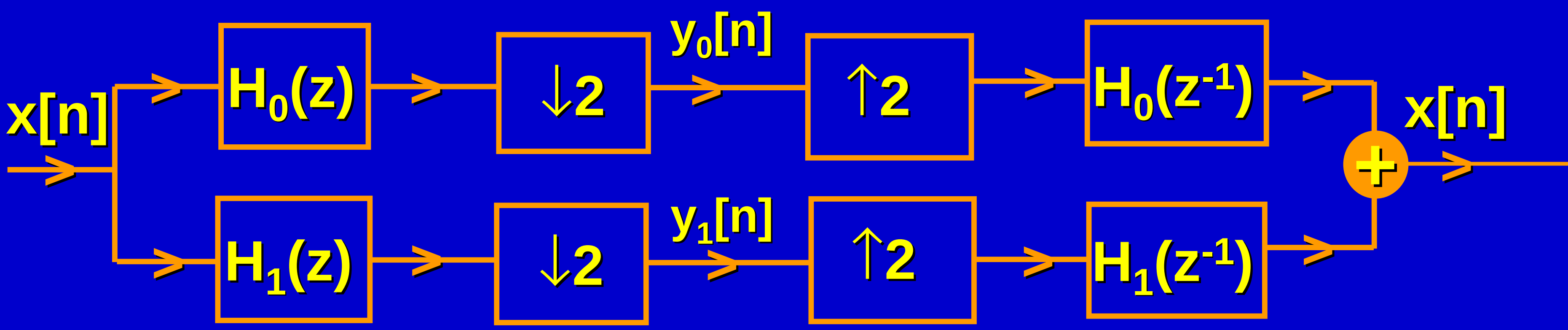
$$H_{1,even}(z) \;=\; z^{(-N+1)/2}\, H_{0,odd}(z^{-1})$$

$$H_{1,odd}(z) \;=\; -z^{(-N+1)/2}\, H_{0,even}(z^{-1})$$

$$\Rightarrow \quad H_{0,even}(z)\, H_{1,even}(z^{-1}) + H_{0,odd}(z)\, H_{1,odd}(z^{-1}) \;=\; 0$$

and $$H_{1,even}(z)\, H_{1,even}(z^{-1}) + H_{1,odd}(z)\, H_{1,odd}(z^{-1}) \;=\; 1$$

Modulation Domain



PR conditions:

$$H_0(z)\,H_0(z^{-1}) + H_1(z)\,H_1(z^{-1}) = 2 \text{ -------(10)}$$

No distortion

$$H_0(-z)\,H_0(z^{-1}) + H_1(-z)\,H_1(z^{-1}) = 0 \text{ --------(11)}$$

Alias cancellation

$$[H_0(z^{-1}) \quad H_1(z^{-1})] \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = [2 \quad 0]$$

1 2 3

$H_m(z)$ **modulation matrix**

Replace z with –z in Equations (10) and (11)

$$H_0(-z)\,H_0(-z^{-1}) + H_1(-z)\,H_1(-z^{-1}) = 2$$

$$H_0(z)\,H_0(-z^{-1}) + H_1(z)\,H_1(-z^{-1}) = 0$$

$$\begin{bmatrix} H_0(z^{-1}) & H_1(z^{-1}) \\ H_0(-z^{-1}) & H_1(-z^{-1}) \end{bmatrix} \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

1 2 3 1 2 3 1 2 3

$H_m^T(z^{-1})$ $H_m(z)$ $2I$

Condition O:

$$H_m^T(z^{-1})\,H_m(z) = 2I \Rightarrow H_m(z) \text{ is paraunitary}$$

Reverse the order of multiplication:

$$\begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix} \begin{bmatrix} H_0(z^{-1}) & H_1(z^{-1}) \\ H_0(-z^{-1}) & H_1(-z^{-1}) \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Express Condition O as a condition on $H_0(z)$:

$$H_0(z)\, H_0(z^{-1}) + H_0(-z)\, H_0(-z^{-1}) = 2 \qquad \text{-----------(12)}$$

Frequency Domain:

$$|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 2 \qquad \text{------------(13)}$$

Again, the remaining conditions are automatically satisfied by the alternating flip choice, $H_1(z) = -z^{-N} H_0(-z^{-1})$

Summary

Condition O as a constraint on the lowpass filter:

- **Matrix form:** $LL^T = I$
- **Coefficient form:** $\sum_n h[n]h[n-2k] = \delta[k]$
- **Polyphase form:**

$$H_{0,even}(z)\, H_{0,even}(z^{-1}) = H_{0,odd}(z)\, H_{0,odd}(z^{-1}) = 1$$

- **Modulation form:** $H_0(z)\, H_0(z^{-1}) + H_0(-z)\, H_0(-z^{-1}) = 2$

Then choose $H_1(z) = -z^{-N} H_0(-z^{-1})$ **; N odd**

i.e., $h_1[n] = (-1)^n h_0[N-n]$