Course 18.327 and 1.130
Wavelets and Filter Banks

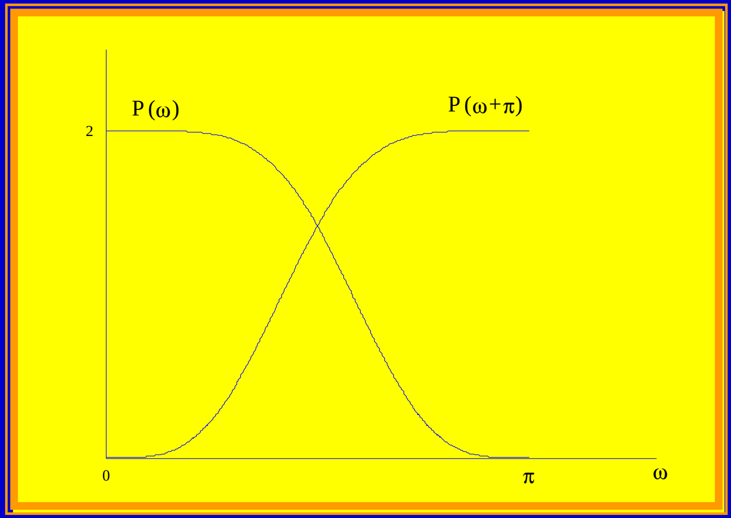
Maxflat Filters: Daubechies and Meyer Formulas.
Spectral Factorization

Formulas for the Product Filter

Halfband condition:

$$P(\omega) + P(\omega + \pi) = 2$$

Also want $P(\omega)$ to be lowpass and $p[n]$ to be symmetric.



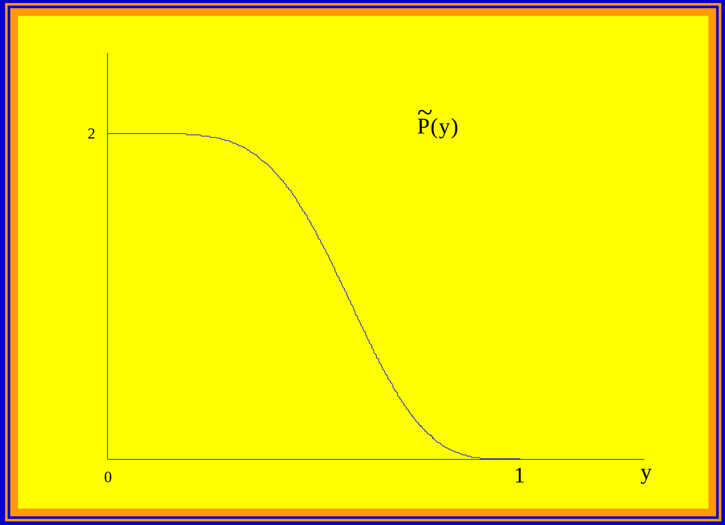
Daubechies' Approach

Design a polynomial, $\tilde{P}(y)$, of degree 2p - 1, such that

$$\tilde{P}(0) = 2$$

$$\tilde{P}^{(l)}(0) = 0; \; l = 1, 2, \ldots, p - 1$$

$$\tilde{P}^{(l)}(1) = 0; \; l = 0, 1, \ldots, p - 1$$



Can achieve required flatness at $y = 1$ by including a term of the form $(1 - y)^p$ i.e.

$$\tilde{P}(y) = 2(1 - y)^p B_p(y)$$

Where $B_p(y)$ is a polynomial of degree $p - 1$.

How to choose $B_p(y)$?

Let $B_p(y)$ be the binomial series expansion for $(1 - y)^{-p}$, truncated after p terms:

$$B_p(y) = 1 + py + \frac{p(p+1)}{2} y^2 + \ldots + \binom{2p-2}{p-1} y^{p-1}$$

$$= (1 - y)^{-p} + O(y^p)$$

< Higher order terms

$$(1-y)^{-1} = \sum_{k=0}^{\infty} y^k$$

$$(1-y)^{-p} = \sum_{k=0}^{\infty} \binom{p+k-1}{k} y^k$$

$$|y| < 1$$

Then

$$\tilde{P}(y) = 2(1-y)^p[(1-y)^{-p} + O(y^p)]$$
$$= 2 + O(y^p)$$

Thus

$$P^{(l)}(0) = 0 \;;\quad l = 1, 2, \ldots, p-1$$

So we have

$$\tilde{P}(y) = 2\,(1-y)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} y^k$$

Now let

$$y = \left(\frac{1 - e^{i\omega}}{2}\right)\left(\frac{1 - e^{-i\omega}}{2}\right) \quad \text{maintains symmetry}$$

$$= \frac{1 - \cos\omega}{2}$$

Thus

$$P(\omega) = \tilde{P}\left(\frac{1 - \cos\omega}{2}\right)$$

$$= 2\left(\frac{1 + \cos\omega}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k+1}{k} \left(\frac{1 - \cos\omega}{2}\right)^k$$

z domain:

$$P(z) = 2\left(\frac{1+z}{2}\right)^p \left(\frac{1+z^{-1}}{2}\right)^p \sum_{k=0}^{p-1} \binom{p+k-1}{k} \left(\frac{1-z}{2}\right)^k \left(\frac{1-z^{-1}}{2}\right)^k$$

Meyer's Approach

Work with derivative of $\tilde{P}(y)$:

$$\tilde{P}'(y) = -C' y^{p-1} (1-y)^{p-1}$$

So

$$\tilde{P}(y) = 2 - C' \int_0^y y^{p-1} (1-y)^{p-1} \, dy \qquad (\tilde{P}(0) = 2)$$

Then

$$P(\omega) = 2 - C' \int_0^\omega \left(\frac{1-\cos\omega}{2}\right)^{p-1} \left(\frac{1+\cos\omega}{2}\right)^{p-1} \frac{\sin\omega}{2} \, d\omega$$

$$= 2 - C' \int_0^\omega \left(\frac{1-\cos^2\omega}{2}\right)^{p-1} \frac{\sin\omega}{2} \, d\omega$$

i.e.

$$P(\omega) = 2 - C \int_0^\omega \sin^{2p-1} \omega \, d\omega$$

Spectral Factorization

Recall the halfband condition for orthogonal filters:

z domain:

$$H_0(z)\,H_0(z^{-1}) + H_0(-z)\,H_0(-z^{-1}) = 2$$

Frequency domain:

$$|H_0(\omega)|^2 + |H_0(\omega + \pi)|^2 = 2$$

The product filter for the orthogonal case is

$$P(z) = H_0(z)\,H_0(z^{-1})$$

$$P(\omega) = |H_0(\omega)|^2 \qquad \Rightarrow P(\omega) \geq 0$$

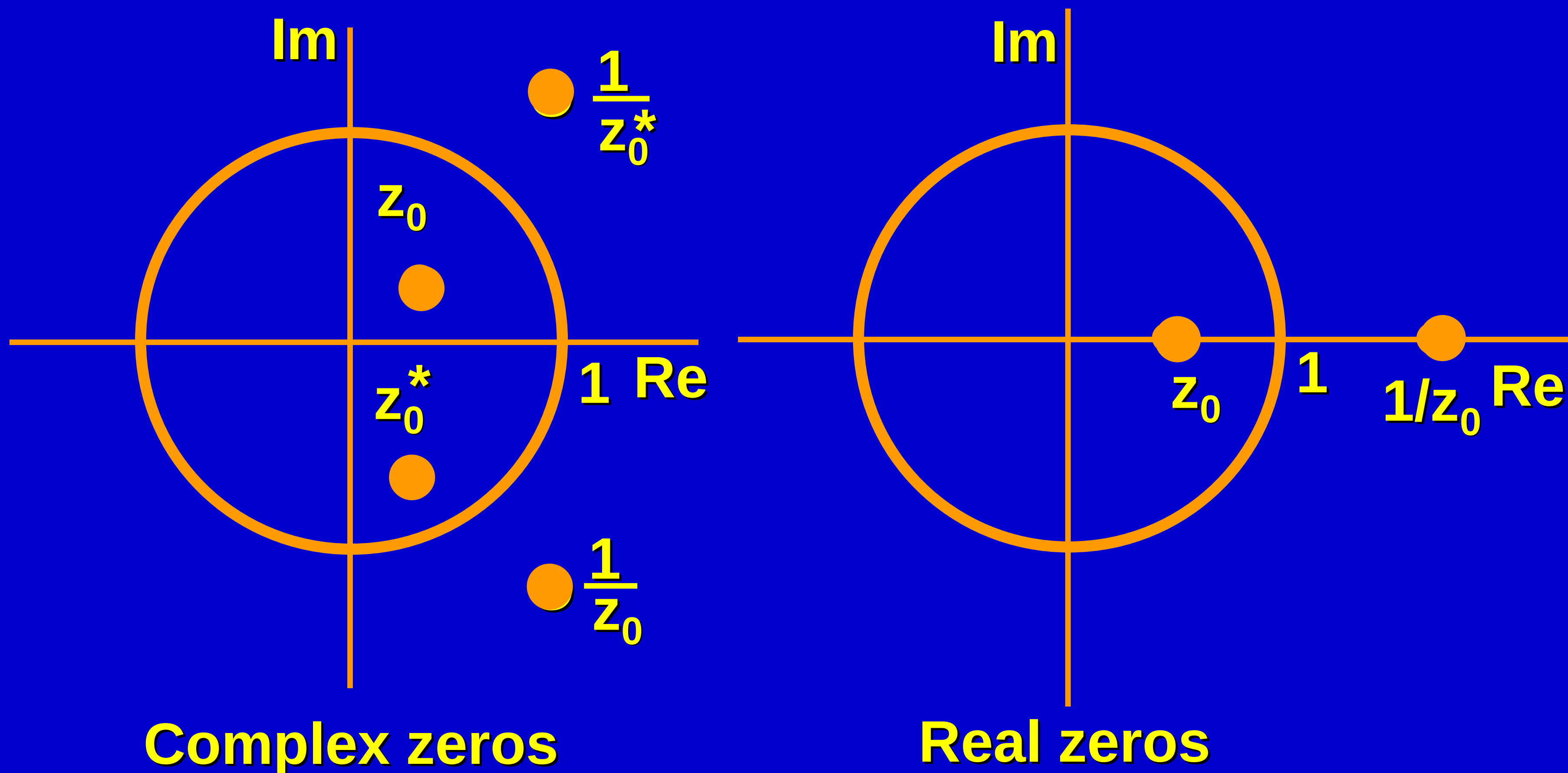
$$p[n] = h_0[n] * h_0[-n] \qquad \Rightarrow p[n] = p[-n]$$

The spectral factorization problem is the problem of finding $H_0(z)$ once $P(z)$ is known.

Consider the distribution of the zeros (roots) of $P(z)$.

- Symmetry of $p[n] \Rightarrow P(z) = P(z^{-1})$
  If z_0 is a root then so is z_0^{-1}.
- If $p[n]$ are real, then the roots appear in complex, conjugate pairs.

$$(1 - z_0 z^{-1})(1 - z_0^* z^{-1}) = 1 - \underbrace{(z_0 + z_0^*)}_{\text{real}} z^{-1} + \underbrace{(z_0 z_0^*)}_{\text{real}} z^{-2}$$



If the zero z_0 is grouped into the spectral factor $H_0(z)$, then the zero $1/z_0$ must be grouped into $H_0(z^{-1})$.

$\Rightarrow$ $h_0[n]$ cannot be symmetric.

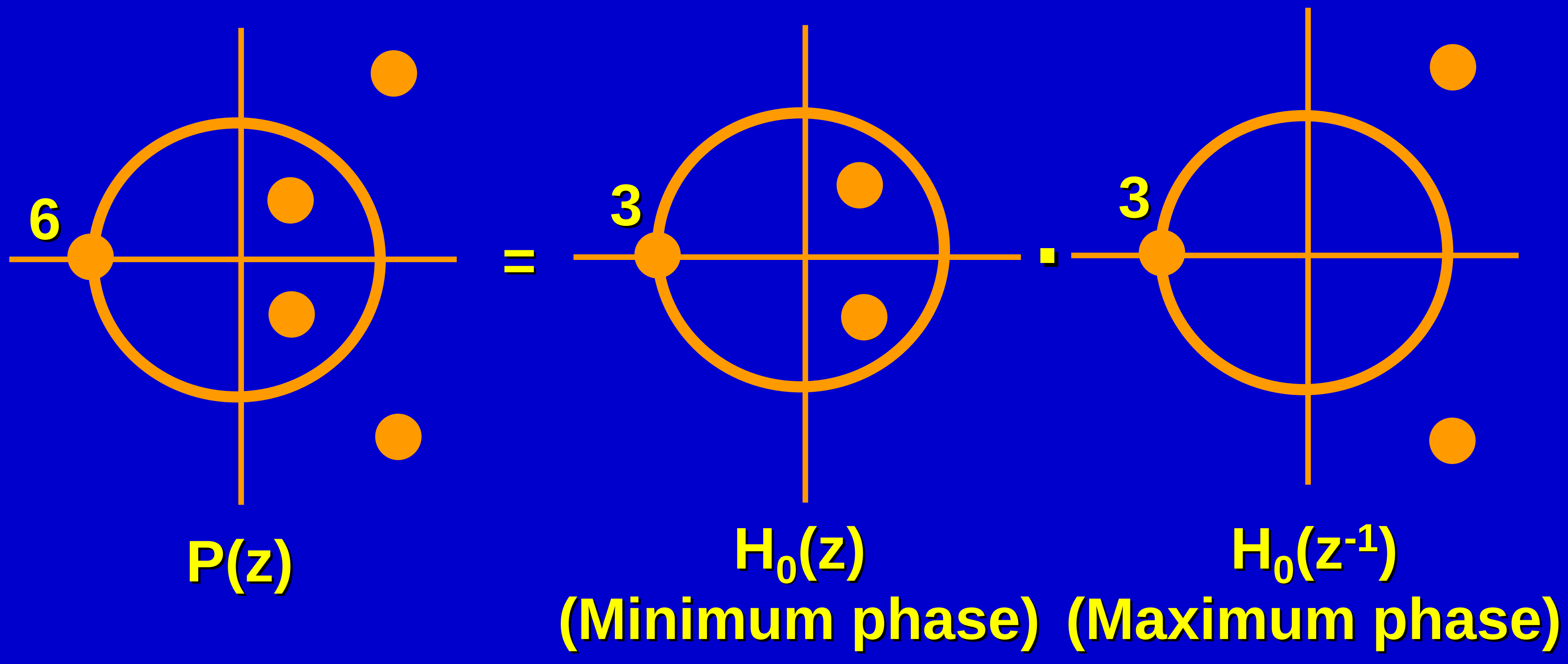
Daubechies' choice: Choose $H_0(z)$ such that

(i) all its zeros are inside or on the unit circle.

(ii) it is causal.

i.e. $H_0(z)$ is a minimum phase filter.

Example:



Practical Algorithms:

1. **Direct Method: compute the roots of P(z) numerically.**
2. **Cepstral Method:**
   First factor out the zeros which lie on the unit circle

$$P(z) = [(1 + z^{-1})(1 + z)]^p \, Q(z)$$

Now we need to factor Q(z) into R(z) R(z^{-1}) such that

i. **R(z) has all its zeros inside the unit circle.**
ii. **R(z) is causal.**

Then use logarithms to change multiplication into addition:

$$Q(z) \quad = \quad R(z) \quad \bullet \quad R(z^{-1})$$

$$\ln Q(z) \quad = \quad \ln R(z) \quad + \quad \ln R(z^{-1})$$

1 2 3 1 2 3 1 2 3

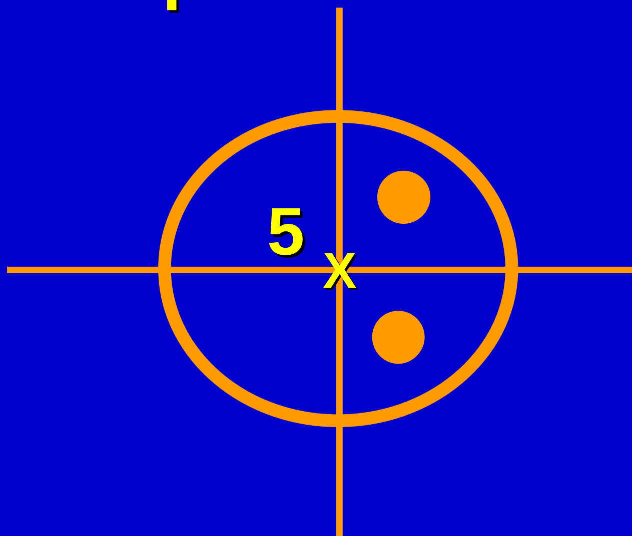
$$\hat{Q}(z) \qquad \hat{R}(z) \qquad \hat{R}(z^{-1})$$

Take inverse z transforms:

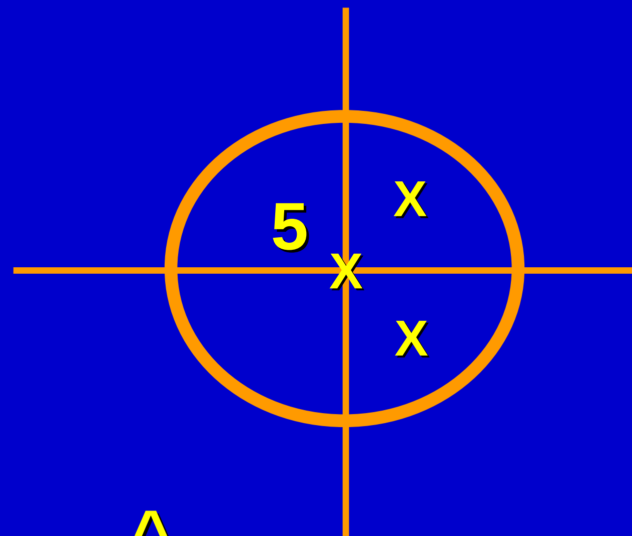
$$\hat{q}[n] \quad = \quad \hat{r}[n] \quad + \quad \hat{r}[-n]$$

Complex cepstrum of q[n]

Example:



$R(z)$



$\hat{R}(z) = \ln R(z)$

$R(z)$ has all its zeros and all its poles inside the unit circle, so $\hat{R}(z)$ has all its singularities inside the unit circle. ($\ln 0 = -\infty$, $\ln \infty = \infty$.)

All singularities inside the unit circle leads to a causal sequence, e.g.

$$X(z) = \frac{1}{1 - z_k\, z^{-1}}$$

Pole at $z = z_k$

$$X(\omega) = \frac{1}{1 - z_k\, ?^{-i\omega}}$$

If $|z_k| < 1$, we can write

$$X(\omega) = \sum_{n=0}^{\infty} (z_k)^n\, e^{-i\omega n}$$

$\Rightarrow x[n]$ is causal

So $\hat{r}[n]$ is the causal part of $\hat{q}[n]$:

$$\hat{r}[n] = \begin{cases} \tfrac{1}{2}\,\hat{q}[0] & ; \quad n = 0 \\ \hat{q}[n] & ; \quad n > 0 \\ 0 & ; \quad n < 0 \end{cases}$$

Algorithm:

Given the coefficients q[n] of the polynomial Q(z):

i. **Compute the M-point DFT of q[n] for a sufficiently large M.**

$$Q[k] = \sum_{n} q[n] e^{-i\frac{2\pi kn}{M}} \quad ; \quad 0 \le k < M$$

ii. **Take the logarithm.**

$$\hat{Q}[k] = \ln(Q[k])$$

iii. **Determine the complex cepstrum of q[n] by computing the IDFT.**

$$\hat{q}[n] = \frac{1}{M} \sum_{k=0}^{M-1} \hat{Q}[k]\, e^{i\frac{2\pi}{M}nk}$$

iv. **Find the causal part of $\hat{q}[n]$.**

$$\hat{r}[n] = \begin{cases} \frac{1}{2}\,\hat{q}[0] & ; \quad n = 0 \\ \hat{q}[n] & ; \quad n > 0 \\ 0 & ; \quad n < 0 \end{cases}$$

v. **Determine the DFT of r[n] by computing the exponent of the DFT of $\hat{r}[n]$.**

$$R[k] = \exp(\hat{R}[k]) = \exp\left(\sum_{k=0}^{M-1} \hat{r}[n] e^{-i\frac{2\pi}{M}kn}\right);\ 0 \le k < M$$

vi. Determine the DFT of $h_0[n]$, by including half the zeros at z = -1.

$$H_0[k] = R[k]\left(1 + e^{-i\frac{2\pi k}{M}}\right)^p$$

vii. Compute the IDFT to get $h_0[n]$.

$$h_0[n] = \frac{1}{M}\sum_{k=0}^{M-1} H_0[k]\, e^{i\frac{2\pi}{M}nk}$$