18.330 :: Homework 5 :: Spring 2012 :: Due Tuesday April 12

1. (4 pts) Consider the initial-value problem for harmonic oscillations: $y''(t) + \omega^2 y(t) = 0$, $y(0) = y_0$, $y'(0) = y_1$. In what follows, consider $y_0 = 0$, $y_1 = 1$, and $\omega = 1$.

   a) Solve the equation explicitly.

   b) Put this second-order equation in the form of a first-order system of two equations, by using the additional unknown $z(t) = y'(t)$.

   c) Solve this system numerically with the forward Euler method. Find a condition on the time step to ensure stability. Plot your solutions as curves in the (y, z) plane.

   d) Same question for the backward Euler method.

   e) Same question for the trapezoidal method.

   f) (Bonus, 1pt) Explain rigorously why the points fall "exactly" on the unit circle for the trapezoidal method.

2. (3 pts) Consider the leap-frog method

$$y_{n+1} = y_{n-1} + 2hf(t_n, y_n).$$

   It is the simplest two-step method for $y' = f(t, y)$. Find its stability zone in the complex hλ-plane.

3. (3 pts) Consider

$$y'(t) = -y^2(t), \qquad y(0) = 100.$$

   The exact solution is $y(t) = 1/(t + 1/100)$.

   a) Prove that the forward Euler iterates y_n obey $y_n \leq 100$.

   b) Find a condition on h to ensure the stability of the forward Euler method. Justify your answer.

   c) Run the forward Euler method on this problem for various time steps and check whether or not the numerical behavior matches the theoretical expectation in part b).

   c) Is it better to use an explicit or an implicit method for this problem? Why?