18.330 :: Homework 7 :: Spring 2012 :: Due Thursday May 3

1. (3pts) Prove the following properties of the Fourier transform ($x, k \in \mathbb{R}$):
   (a) Dilation: if $g(x) = f(x/a)$ for $a > 0$, then $\hat{g}(k) = a\hat{f}(ak)$.
   (b) Conjugation: if $g(x) = \overline{f}(x)$, then $\hat{g}(k) = \overline{\hat{f}(-k)}$.
   (c) Symmetry: if f is real and even $(f(x) = f(-x))$, show that $\hat{f}(k)$ is also real and even.
   (d) Symmetry: if f is real and odd $(f(x) = f(-x))$, show that $\hat{f}(k)$ is imaginary and odd.

2. (3pts) Consider the sinc function
$$\text{sinc}(x) = \frac{\sin x}{x}.$$
   (a) Show that sinc is not integrable, i.e. sinc $\notin L^1(\mathbb{R})$. [Hint: if $f(x) \geq g(x) \geq 0$ and $\int g(x)dx$ diverges, then $\int f(x)dx$ diverges as well.]
   (b) Nevertheless, integrals involving sinc may make sense. From the theory of Fourier transforms, predict the value of
$$\int_{-\infty}^{\infty} \text{sinc}(x)dx.$$

3. (2pts) Consider a kernel $K(x)$, and the integral equation
$$u(x) + \int_{-\infty}^{\infty} K(x-y)u(y)dy = f(x).$$
   Similar looking equations arise for instance in rendering in computer graphics, and in the scattering of radar waves off of planes. Find a formula for the solution $u(x)$ of the above equation, using the Fourier transform. What is the condition on $K(x)$ such that no division by zero occurs?

4. (2pts) Show that the functions
$$v_k(x) = ce^{-ikx}, \qquad k \in \mathbb{Z}$$
   are orthogonal over $[0, 2\pi]$, for the inner product $\langle f, g \rangle = \int_0^{2\pi} f(x)\overline{g}(x)dx$. Find the value of c such that these functions are also normalized.