Smallest Eigenvalue of M-Matrices

- Def: M-Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix};$$

$a_{ii} \geq 0$

$a_{ij} \leq 0$, $i \neq j$

Row Sums $s_i = \Sigma_{j=1}^{n} a_{ij} \geq 0$

- Given: Row sums s_i and off diagonals $a_{ij}, i \neq j$.
- Diagonal elements computable accurately, sum of positives

$$a_{ii} = s_i - \sum_{j \neq i} a_{ij}$$

GE on Weakly Diagonally Dominant M-Matrices

- Pivoting, if needed, is diagonal, preserves structure
- One step of GE:
  - Off diagonals: $a_{ij} = a_{ij} - \frac{a_{ik}a_{kj}}{a_{kk}}$
  - Row sums: $s_i = s_i - \frac{a_{ik}}{a_{kk}} s_k$
- Everything is preserved in Schur complementation
  - Weak diagonal dominance
  - M-matrix structure
  - High relative accuracy in a_{ij} and s_i
- Yields Cholesky factors

Getting the inverse

- Again no subtractions in solving

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} \\ & c_{22} & c_{23} & c_{24} \\ & & c_{33} & c_{44} \\ & & & c_{44} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

- Think of b as e_i or > 0 in general.

$$\begin{aligned} x_4 &= b_4/c_{44} \\ x_3 &= (b_3 - c_{34}x_4)/c_{33} \\ x_2 &= (b_2 - c_{24}x_4 - c_{23}x_3)/c_{22} \\ x_1 &= (b_1 - c_{14}x_4 - c_{13}x_3c_{12}x_2)/c_{11} \end{aligned}$$

- Solving with C^T analogous $\Rightarrow A^{-1}$ – positive.
- Accurate (Positive) Inverse = Accurate smallest eigenvalue (even in the nonsymmetric case)