2 Homework Solutions

18.335 - Fall 2004

2.1 Count the number of floating point operations required to compute the QR decomposition of an m-by-n matrix using (a) Householder reflectors (b) Givens rotations.

(a) See Trefethen p. 74-75. Answer: $\sim 2mn^2 - \frac{2}{3}n^3$ flops.

(b) Following the same procedure as in part (a) we get the same 'volume', namely $\sim \frac{1}{2}mn^2 - \frac{1}{6}n^3$. The only difference we have here comes from the number of flops required for calculating the Givens matrix. This operation requires 6 flops (instead of 4 for the Householder reflectors) and hence in total we need $\sim 3mn^2 - n^3$ flops.

2.2 Trefethen 5.4

Let the SVD of $A = U\Sigma V^*$. Denote with v_i the columns of V , u_i the columns of U and σ_i the singular values of A. We want to find $x = (x_1; x_2)$ and λ such that:

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

This gives $A^*x_2 = \lambda x_1$ and $Ax_1 = \lambda x_2$. Multiplying the 1st equation with A and substitution of the 2nd equation gives $AA^*x_2 = \lambda^2 x_2$. From this we may conclude that x_2 is a left singular vector of A. The same can be done to see that x_1 is a right singular vector of A. From this the $2m$ eigenvectors are found to be:

$$x_{\pm} = \frac{1}{\sqrt{2}} \begin{pmatrix} v_i \\ \pm u_i \end{pmatrix}, \ i = 1...m$$

corresponding to the eigenvalues $\lambda = \pm\sigma_i$. Therefore we get the eigenvalue decomposition:

$$\begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} V & V \\ U & -U \end{pmatrix} \begin{pmatrix} \Sigma & 0 \\ 0 & -\Sigma \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} V & V \\ U & -U \end{pmatrix}^{-1}$$

2.3 If $A = R + uv^*$, where R is upper triangular matrix and u and v are (column) vectors, describe an algorithm to compute the QR decomposition of A in $\mathcal{O}(n^2)$ time.

The matrix A is of the form

$$A = \begin{pmatrix} * & * & * & \cdots & * & * \\ u_2v_1 & * & & & & * \\ u_3v_1 & u_3v_2 & * & & & \vdots \\ u_4v_1 & u_4v_2 & u_4v_3 & * & & \vdots \\ \vdots & & & & \ddots & * \\ u_nv_1 & u_nv_2 & \dots & \cdots & u_nv_{n-1} & * \end{pmatrix}$$

We exploit the fact that the matrix uv^* is rank one. By applying a sequence of Givens rotations starting from the bottom row, we notice that the rotation that zeroes the entry $A_{k,1}$ also zeroes out all the entries $A_{k,2}$, $A_{k,3}, ...A_{k,2n-k-2}$. Thus the $n-1$ Givens rotations that kill the first column also kill all the entries below the first subdiagonal:

$$\begin{pmatrix} * & * & \cdots & * & * \\ \times & * & & & * \\ 0 & \times & * & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \times & * \end{pmatrix}$$

Thus we need another $n-1$ Givens rotations to kill the first subdiagonal entries (shown with $\times$'s above). We have a total cost $2n-2$ rotations at no more than $6n$ operations per Givens rotation. Hence this algorithm requires $\mathcal{O}(n^2)$ flops.

2.4 Given the SVD of A, compute the SVD of $(A^*A)^{-1}$, $(A^*A)^{-1}A^*$, $A(A^*A)^{-1}$, $A(A^*A)^{-1}A^*$ in terms of U, Σ and V.

Answers:

- $(A^*A)^{-1} = V\Sigma^{-2}V^*$
- $(A^*A)^{-1}A^* = V\Sigma^{-1}U^*$
- $A(A^*A)^{-1} = U\Sigma^{-1}V^*$
- $A(A^*A)^{-1}A^* = UU^*$ (note that UU^* may not be equal to I, unless U is square in the reduced SVD)