8.4 Cholesky Factorization

Let A be Hermitian, positive definite ($A = A^*$ and $x^* Ax \geq 0$, $\forall x$), then by exploring symmetry GE costs $\frac{1}{3}n^3$ and there is no need for pivoting.

$$\begin{aligned} A &= R^* R && (8.70) \\ \|R\| &= \|A\|^{\frac{1}{2}} && (8.71) \\ A + \delta A &= \tilde{R}^* \tilde{R} \\ &= \frac{\|\delta A\|}{\underbrace{\|R\| \cdot \|R^*\|}_{\|A\|}} \\ &= O(\epsilon) && (8.72) \end{aligned}$$

Always backward stable.

A matrix A is *symmetric positive definite (s.p.d.)* if it is symmetric ($A^T = A$),

$$x^T Ax \geq 0 \quad \text{for every } x, \text{ and } x^T Ax = 0 \quad \text{only when } x = 0 \tag{8.73}$$

If A is s.p.d. then all eigenvalues of A are positive and all leading principal minors $A(1 : k, 1 : k) > 0$, $k = 1, 2, \ldots, n$.

Let

$$A = LDU \tag{8.74}$$

be the LDU decomposition of an s.p.d. matrix A. Then

$$U^T D^T L^T = U^T DL^T = A^T = A = LDU. \tag{8.75}$$

Since U^T and L^T are unit lower and upper triangular matrices respectively, we obtain two LDU decompositions of A:

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$$A = U^T D L^T, \tag{8.76}$$

and

$$A = LDU. \tag{8.77}$$

These decompositions are the same, therefore $U^T = L$. Finally,

$$A = LDL^T. \tag{8.78}$$

We can also write $A = LDL^T = (LD^{1/2})(D^{1/2}L^T) = CC^T$, where $C = LD^{1/2}$ is lower triangular (all elements of D, $d_i = \frac{\det A(1{:}k,1{:}k)}{\det A(1{:}k-1,1{:}k-1)} > 0$, so we can safely form $D^{1/2}$).

Definition: The decomposition

$$A = CC^T \tag{8.79}$$

of an s.p.d. matrix as a product of a nonsingular lower triangular matrix and its transpose is called *Cholesky decomposition*.

Theorem: A matrix A is s.p.d. if and only if it has a Cholesky decomposition.

Proof: If A is s.p.d. then it has a Cholesky decomposition as we described above.

If $A = CC^T$, where C is nonsingular, let $y = C^T x$, then

$$x^T A x = x^T C C^T x = (C^T x)^T C^T x = y^T y = y_1^2 + y_2^2 + \cdots + y_n^2 \geq 0 \tag{8.80}$$

with equality only when $y = 0$, i.e., only when $x = 0$ (since $y = C^T x$, and C^T is nonsingular).

Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 13 & 18 \\ 3 & 18 & 50 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 2 & 3 & \\ 3 & 4 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ & 3 & 4 \\ & & 5 \end{bmatrix} \tag{8.81}$$

Exercise: If A is s.p.d., then $a_{ii}a_{jj} > |a_{ij}^2|$, $i \neq j$.