Chapter 10

10.3 Simultaneous Iterations

$$V = [v_1, \cdots, v_n] \tag{10.1}$$

$$V^{(i)} = AV^{(i-1)} \tag{10.2}$$

Instead take

$Q^{(0)} = [q_1, \cdots, q_n]$-`orthogonal`
`for` $k = 1, 2, \cdots$

$Z = AQ^{(k-1)}$
$Q^{(k)}R^{(k)} = Z$
$A^{(k)} = (Q^{(k)})^T AQ^{(k)}$

Chapter 11

11.1 Simultaneous Iteration

$A^{(k)}V \rightarrow$ problematic, all columns of V converge to same v_1. Orthogonalized V:

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Pick Q^(0)--orthogonal
For k = 1, 2, ...
  Z = A Q̲^(k-1)
  Q̲^(k) R^(k) = Z
Define A^(k) = (Q̲^(k))^T A Q̲^(k)
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11.2 QR Iteration

$$A^{(0)} = A \tag{11.1}$$
$$A^{(k-1)} = Q^{(k)}R^{(k)} \tag{11.2}$$
$$A^{(k)} = R^{(k)}Q^{(k)} \tag{11.3}$$
$$\underline{Q^{(k)}} = Q^{(1)} \cdots Q^{(k)} \tag{11.4}$$

Define

$$\underline{R^{(k)}} = R^{(k)} \cdots R^{(1)} \tag{11.5}$$

Therefore, $A^{(k)}$ are the same.

$$A^{(k)} = \underline{Q^{(k)}}\,\underline{R^{(k)}} \tag{11.6}$$

Proof: Induction.

$$\begin{aligned} A^{(k)} &= A\underline{Q}^{(k-1)}\underline{R}^{(k-1)} \\ &= \underline{Q}^{(k)}R^{(k)}\underline{R}^{(k-1)} \\ &= \underline{Q}^{(k)}\underline{R}^{(k)}. \end{aligned} \tag{11.7}$$

Because of independent assumption on

$$A^{(k-1)} = (Q^{(k-1)})^T A Q^{(k-1)}. \tag{11.8}$$

Formally:

$$\begin{aligned} A^{(k)} &= (Q^{(k)})^T A^{(k-1)} Q^{(k)} \\ &= \underline{(Q^{(k)})^T} A^{(k)} \underline{Q^{(k)}}. \end{aligned} \tag{11.9}$$

From QR:

$$\begin{aligned} A^{(k)} &= A\underline{Q}^{(k-1)}\underline{R}^{(k-1)} \\ &= \underline{Q}^{(k-1)}A^{(k-1)}\underline{R}^{(k-1)} \\ &= \underline{Q}^{(k)}\underline{R}^{(k)}. \end{aligned} \tag{11.10}$$

11.3 Connection with Inverse Iteration

Let A be real and symmetric.

$$A^{(k)} = \underline{Q}^{(k)}\underline{R}^{(k)} \tag{11.11}$$

$$\begin{aligned} (A^k)^{-1} &= (R^{(k)})^{-1}(Q^{(k)})^T \\ &= Q^{(k)}(R^{(k)})^{-T}. \end{aligned} \tag{11.12}$$

Let P be the reverse identity

$$P_{ij} = (\delta_{n+1-i,j})_{i,j=1}^n \tag{11.13}$$

$$P = \begin{bmatrix} & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & \end{bmatrix} \tag{11.14}$$

$$A^{-1}P = (\underline{Q}^{(k)}P)[P(\underline{R}^{(k)})^{-T}P] \tag{11.15}$$

Simultaneous iteration on P. Simultaneous inverse iteration on A.

11.4 Shifts in QR

$$\begin{aligned}
A^{(k-1)} - \mu^{(k)} I &= Q^{(k)} R^{(k)} && (11.16)\\
A^{(k)} &= R^{(k)} Q^{(k)} + \mu^{(k)} I \\
&= (Q^{(k)})^T A^{(k-1)} Q^{(k)} \\
&= \cdots \\
&= (\underline{Q^{(k)}})^T A \underline{Q^{(k)}} && (11.17)\\
(A-\mu^{(k)} I)(A-\mu^{(k-1)} I)\cdots(A-\mu^{(1)} I) &= Q^{(k)} R^{(k)} && (11.18)\\
\underline{Q^{(k)}} &= \prod_{j=1}^{k} Q^{(j)} && (11.19)
\end{aligned}$$

is an orthogonalization of A.

11.5 Choice of Shifts

Converge to the last column of $Q^{(k)}$.

Shift:

$$\begin{aligned}
\mu^{(k)} &= \frac{(q_n^{(k)})^T A q_n^{(k)}}{(q_n^{(k)})^T q_n^{(k)}} \\
&= (q_n^{(k)})^T A q_n^{(k)} \\
&= (Q^{(k)} e_n)^T A (Q^{(k)} e_n) \\
&= e_n^T (Q^{(k)})^T A (Q^{(k)}) e_n \\
&= e_n^T A^{(k)} e_n \\
&= A_{nn}^{(k)} && (11.20)
\end{aligned}$$

Rayleigh quotient shift with starting vector e_n.

Rayleigh quotient shift does not break this matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Need Wilkinson shift $\to$ eigenvalue of $\begin{bmatrix} a_{m-1} & b_{m-1} \\ b_{m-1} & a_m \end{bmatrix}$ closer to a_m.

11.6 Stability

QR iteration is backward stable.

$$\tilde{Q}\tilde{\Lambda}\tilde{Q} = A + \delta A \qquad (11.21)$$

$$
\begin{aligned}
\frac{\|\delta A\|}{\|A\|} &= O(\epsilon) &(11.22)\\
|\tilde{\lambda}_j - \lambda_j| &= O(\epsilon)\,\|A\| &(11.23)
\end{aligned}
$$

11.7 Jacobi Method

Algorithm for finding eigenvalues of symmetric matrices.

Idea: Form J^TAJ, J–orthogonal, in such a way that $\|\cdot\|_F$ is preserved, but $\text{off}(A)$ is reduced the off diagonal, where $\text{off}(A) = \sum_{i\neq j} |a_{ij}|^2$ is the sum of the squares of the off diagonal elements.

$$
\begin{aligned}
J^T \begin{bmatrix} a & d \\ d & b \end{bmatrix} J &= \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} &(11.24)\\
J &= \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} &(11.25)\\
\tan 2\theta &= \frac{2d}{b-a} &(11.26)
\end{aligned}
$$

Which a_{ij} do we pick at every step? Pick largest, therefore $\text{off}(A)$ decreases by a factor of $1 - \frac{2}{m^2-m}$. Therefore, $O(m^2)$ steps ($O(m)$ operations per step) are needed for convergence.