Chapter 12

12.1 Bisection

Idea: Find all eigenvalues in an interval.

12.1.1 Tridiagonal Form

$$A = \begin{bmatrix} a_1 & b_1 & & \\ b_1 & \ddots & \ddots & \\ & \ddots & \ddots & b_{n-1} \\ & & b_{n-1} & a_n \end{bmatrix} \tag{12.1}$$

Identify roots of $p(x) = (A - xI)$, all roots distinct.
Define $A^{(k)} = A_{1:k,1:k}$ leading principal sub-matrices.
Eigenvalues of $A^{(k)}$: $\lambda_j^{(k)}$, $k = 1, \cdots, n$, $j = 1, \cdots, k$. Ordered $\lambda_1^{(k)} < \cdots < \lambda_k^{(k)}$.

$$\det(A^{(k)}) = a_k \det(A^{(k-1)}) - b_{k-1}^2 \det(A^{(k-2)}) \tag{12.2}$$

$$p^{(k)}(x) \equiv \det(A^{(k)} - xI)$$

$$= (a_k x) p^{(k-1)}(x) - b_{k-1}^2 p^{(k-2)}(x) \tag{12.3}$$

Eigenvalues of $A^{(k)}$ and $A^{(k+1)}$ interlace

$$\lambda_j^{(k+1)} < \lambda_j^{(k)} < \lambda_{j+1}^{(k+1)} \tag{12.4}$$

Consider $A = \begin{bmatrix} 1 & 1 & & \\ 1 & 0 & 1 & \\ & 1 & 2 & 1 \\ & & 1 & -1 \end{bmatrix}$

$$\det(A^{(1)}) = 1 \tag{12.5}$$
$$\det(A^{(2)}) = -1 \tag{12.6}$$
$$\det(A^{(3)}) = -3 \tag{12.7}$$
$$\det(A^{(4)}) = 4 \tag{12.8}$$

Therefore, $A^{(1)}$ has no negative eigenvalues, $A^{(2)}$ has one negative eigenvalue, $A^{(3)}$ has one negative eigenvalue, and $A^{(4)}$ has two negative eigenvalues.

$$\#\text{ of negative eigenvalues} = \#\text{ of sign changes in } \underbrace{1, \det(A^{(1)}), \cdots, \det(A^{(n)})}_{\text{Strum sequence}} \tag{12.9}$$

Same argument to a shifted matrix. $A - xI \Rightarrow$ eigenvalues in any interval. Cost $O(m \log \epsilon)$ per eigenvalue.

12.2 Divide and Conquer

$$\begin{aligned}
T &= \left[\begin{array}{c|c} T_1 & \beta \\ \hline \beta & T_2 \end{array}\right] \\
&= \left[\begin{array}{c|c} \tilde{T}_1 & 0 \\ \hline 0 & \tilde{T}_2 \end{array}\right] + \left[\begin{array}{c|c} \beta & \beta \\ \hline \beta & \beta \end{array}\right] \\
&= \left[\begin{array}{c|c} \tilde{T}_1 & \\ \hline & \tilde{T}_2 \end{array}\right] + \beta Z Z^T
\end{aligned} \tag{12.10}$$

$$Q^T T Q = \left[\begin{array}{c|c} D_1 & \\ \hline & D_2 \end{array}\right] + WW^T \tag{12.11}$$

Eigenvalues of $D + WW^T$

$$\det(I + xy^T) = 1 + x^T y \tag{12.12}$$

$$\det(D + WW^t - \lambda I) = 0 \tag{12.13}$$

$$\det(\underbrace{(D - \lambda I)}_{assume\ non\ zero} (I + (D - \lambda I)^{-1} WW^T)) = 0 \tag{12.14}$$

$$1 + W^T (D - \lambda I)^{-1} W = 0 \tag{12.15}$$

$$1 + \sum_{j=1}^{n} \frac{W_j^2}{d_j - \lambda} = 0 \ secular\ equation \tag{12.16}$$

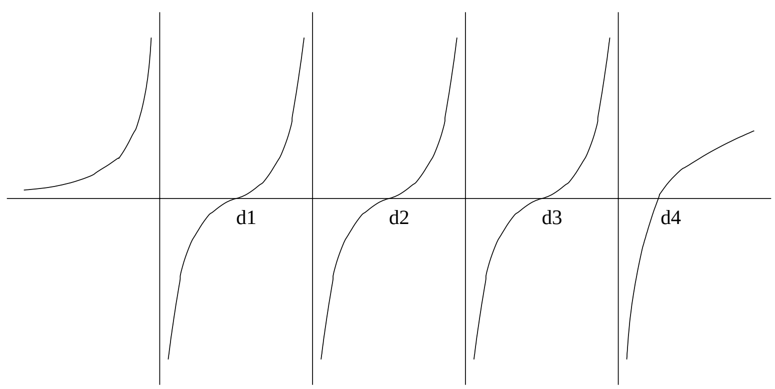

Figure 12.1: Solving the Secular Equation.

Cost $O(m)$ per root $O(m^2)$.

$$\begin{aligned} Total &= O\left(m^2 + 2\left(\frac{m}{2}\right)^2 + 4\left(\frac{m}{4}\right)^2 + \cdots\right) \\ &= O(m^2) \end{aligned} \tag{12.17}$$