# Chapter 15

## 15.1 Conjugate Gradients (CG)

Minimize

$$\|r_k\|_{A^{-1}}, \tag{15.1}$$

where $r_k = b - Ax_k$. Also means $\|e_k\|_A$ is minimized, where $e_k = x_* - x_k$.

$$\begin{aligned} x_0 &= 0 && (15.2)\\ r_0 &= b && (15.3)\\ p_0 &= r_0 && (15.4) \end{aligned}$$

For $k = 1, 2, 3, \cdots$

$$\begin{aligned} \text{step length } \alpha_k &= \frac{r_{k-1}^T r_{k-1}}{p_{k-1}^T A p_{k-1}} && (15.5)\\ \text{improve solution } x_k &= x_{k-1} + \alpha_k p_{k-1} && (15.6)\\ \text{residual } r_k &= r_{k-1} - \alpha_k A p_{k-1} && (15.7)\\ \text{improvement } \beta_k &= \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}} && (15.8)\\ \text{search dir. } p_k &= r_k + \beta_k p_{k-1} && (15.9) \end{aligned}$$

**Theorem:** Let $A$ be s.p.d. If $r_{n-1} \neq 0$ (no convergence yet), then:

1.

$$K_k = \langle x_1, \cdots, x_k \rangle$$

$$
\begin{aligned}
&= \langle p_0, \cdots, p_{k-1} \rangle \\
&= \langle r_0, \cdots, r_{k-1} \rangle \\
&= \langle b, Ab, \cdots, A^{k-1}b \rangle
\end{aligned} \tag{15.10}
$$

2.

$$
r_k^T r_j = 0 \quad k > j \tag{15.11}
$$

3.

$$
p_k^T A p_j = 0 \quad k > j \tag{15.12}
$$

($A$-conjugate search dir.)

**Proof:**

1.

$$
x_0 = 0 \tag{15.13}
$$

$$
x_k = x_{k-1} + \alpha_k p_{k-1} \tag{15.14}
$$

therefore,

$$
x_k \in \langle p_0, \cdots, p_{k-1} \rangle \tag{15.15}
$$

and vice versa.

$$
p_k = r_k + p_k p_{k-1} \tag{15.16}
$$

therefore,

$$
\langle p_0, \cdots, p_{k-1} \rangle = \langle r_0, \cdots, r_{k-1} \rangle \tag{15.17}
$$

$$
r_k = r_{k-1} - \alpha_k A p_{k-1} \tag{15.18}
$$

therefore,

$$
\langle r_0, \cdots, r_{k-1} \rangle = \langle b, Ab, \cdots, A^{k-1}b \rangle \tag{15.19}
$$

2.

$$
r_k = r_{k-1} - \alpha_k A p_{k-1} \tag{15.20}
$$

$$
r_k^T r_j = r_{k-1}^T r_j - \alpha_k p_{k-1}^T A r_j \tag{15.21}
$$

If $j < k-1$ then $r_k^T r_j = 0$ by reduction. If $j = k-1$ we need

$$\alpha_k = \frac{r_{k-1}^T r_{k-1}}{p_{k-1}^T A r_{k-1}} \tag{15.22}$$

We have

$$\alpha_k = \frac{r_{k-1}^T r_{k-1}}{p_{k-1}^T A p_{k-1}} \tag{15.23}$$

$$\begin{aligned} p_{k-1} &= r_{k-1} + \beta_{k-1} p_{k-2} && (15.24) \\ p_{k-1}^T A p_{k-1} &= p_{k-1}^T A r_{k-1} + p_{k-1} \underbrace{p_{k-1}^T A p_{k-2}}_{=0} && (15.25) \end{aligned}$$

3. Use

$$\begin{aligned} p_k &= r_k + \beta_k p_{k-1} && (15.26) \\ p_K^T A p_j &= r_k^T A p_j + \beta_k p_{k-1}^T A p_k && (15.27) \end{aligned}$$

If $j < k-1$ then $p_k^T A p_j = 0$ by reduction. If $j = k-1$ we need

$$\begin{aligned} \beta_k &= -\frac{r_k^T A p_{k-1}}{p_{k-1}^T A p_{k-1}} \\ &= -\frac{\alpha_k r_k^T A p_{k-1}}{\alpha_k p_{k-1}^T A p_{k-1}} && (15.28) \end{aligned}$$

Questions:

$$\begin{aligned} r_k^T(-\alpha A p_{k-1}) &\stackrel{?}{=} r_k^T r_k \quad \text{yes} && (15.29) \\ r_{k-1}^T r_{k-1} &\stackrel{?}{=} p_{k-1}^T(\alpha A p_{k-1}) \quad \text{yes} && (15.30) \end{aligned}$$