3.4 QR Factorization

Idea: We have to solve $Ax = b$:

If A is square: $x = A^{-1}b$. If A is rectangular, we want to find $x : \min_x \|Ax - b\|_2$.

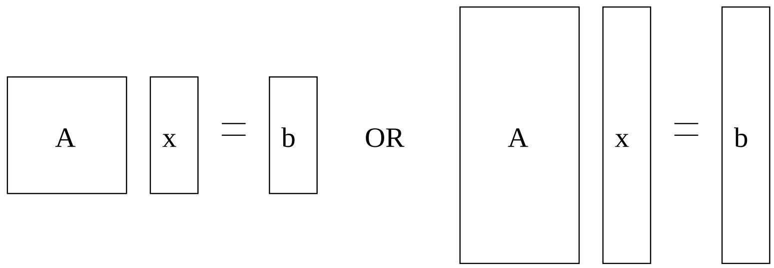

Figure 3.1: QR Factorization.

What linear systems can we solve easily?

- Triangular: $Rx = b$

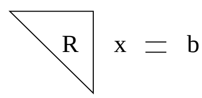

Figure 3.2: Triangular.

- Orthogonal: $Qx = b \Rightarrow x = Q^{-1}b = Q^{*}b$ (easy)

(Continued on next page.)

Write $A = QR$, then solve $QRx = b \Rightarrow Rx = Q^*b$.

$$x = R^{-1}Q^*b \tag{3.6}$$

Solves linear systems and least square problems.

3.5 What is QR Decomposition?

$$A = [a_1|a_2|\cdots|a_n] \tag{3.7}$$

Idea: Find orthogonal q_i:

$$\text{span}\langle a_1, \cdots, a_i\rangle = \text{span}\langle q_1, \cdots, q_i\rangle \tag{3.8}$$

for $i = 1, 2, \cdots, n$.

$$a_1 = r_{11}q_1 \tag{3.9}$$
$$a_2 = r_{12}q_1 + r_{22}q_2 \tag{3.10}$$
$$\cdots$$
$$a_n = r_{1n}q_1 + \cdots + r_{nn}q_n \tag{3.11}$$

Therefore,

$$A = \underbrace{[q_1|\cdots|q_n]}_{Q} \underbrace{\begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \\ & & r_{nn} \end{bmatrix}}_{R} \tag{3.12}$$

How do we find Q and R?

3.6 Gram-Schmidt

$$q_1 = \frac{a_1}{\|a_1\|} = \frac{a_1}{r_{11}} \tag{3.13}$$
$$q_2 = \frac{a_2 - r_{12}q_1}{\|a_2 - r_{12}q_1\|} = \frac{a_2 - r_{12}q_1}{r_{22}} \tag{3.14}$$
$$q_3 = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{\|a_3 - r_{13}q_1 - r_{23}q_2\|} = \frac{a_3 - r_{13}q_1 - r_{23}q_2}{r_{33}} \tag{3.15}$$
$$\cdots$$
$$q_n = \frac{a_n - \sum_{i=1}^{n-1} r_{in}q_i}{r_{nn}} \tag{3.16}$$

How do we determine r_{ij}?

$$q_i^* q_j = 0 \qquad i \neq j \tag{3.17}$$

$$0 = q_i^* q_j - r_{ij} \qquad i \neq j \tag{3.18}$$

$$r_{ij} = \left\| a_j - \sum_{i=1}^{j-1} r_{ij} q_i \right\|_2 \tag{3.19}$$

Classical Gram-Schmidt is numerically unstable. We will use two other algorithms: Givens rotations and Householder reflectors.