18.335 Problem Set 3

Problem 1: SVD and low-rank approximations

(a) Show that for the $A = \hat{Q}\hat{R}$ decomposition from Trefethen chapter 7, A and $\hat{R}$ have the same singular values.

(b) Trefethen, probem 4.5.

(c) Trefethen, problem 5.2.

(d) Take any grayscale photograph (either one of your own, or off the web). Scale it down to be no more than 1500×1500 (but not necessarily square), and read it into Matlab as a matrix A with the `imread` command [type "`doc imread`" for instructions: in particular you'll want to use a command like `A = double(imread('myfile.jpg'));`]. (Color images are more complicated because they have red/green/blue components; I would stick with a grayscale image.)

  (i) Compute the SVD of A (Matlab's svd command) and plot the singular values (e.g. as a histogram, possibly on a log scale) to show the distribution.

  (ii) Compute a lower-rank approximation of A by taking only the largest v singular values for some v (as in theorem 5.8). You can save this approximation as an image using `imwrite`, or you can plot it directly using the `pcolor` command [`pcolor(flipud(A)); colormap(gray); shading interp; axis equal`]. How big does v have to be to get a reasonably recognizable image?

Problem 2: QR and orthogonal bases

(a) Prove that $A = QR$ and $B = RQ$ have the same eigenvalues, assuming A is a square matrix. Construct a random 10×10 real-symmetric matrix in Matlab via `X=rand(10,10); A = X' * X`. Use `[Q,R] = qr(A)` to compute the QR factorization of A, and then compute $B = RQ$. Then find the QR factorization $B = Q'R'$, and compute $R'Q'$...repeat this process until the matrix converges. From what it converges to, suggest a procedure to compute the eigenvalues and eigenvectors of a matrix (no need to prove that it converges in general).

(b) Trefethen, problem 7.2.

(c) Trefethen, problem 10.4.

Problem 3: Schur fine

In class, we showed that any square $m \times m$ matrix A can be factorized as $A = QTQ^*$, where Q is unitary and T is an upper-triangular matrix (with the same eigenvalues as T).

1. A is called "normal" if $AA^* = A^*A$. Show that this implies $TT^* = T^*T$. From this, show that T must be diagonal. Hence, any normal matrix (e.g. unitary or Hermitian matrices) must be unitarily diagonalizable. Hint: consider the diagonal entries of TT^* and T^*T, starting from the (1,1) entries and proceeding diagonally downwards by induction.

2. Given the Schur factorization of an arbitary A (not necessarily normal), describe an algorithm to find the eigenvalues and eigenvectors of A, assuming for simplicity that all the eigenvalues are distinct. The flop count should be asymptotically $Km^3 + O(m^2)$; give the constant K.

18.335J / 6.337J Introduction to Numerical Methods
Fall 2010