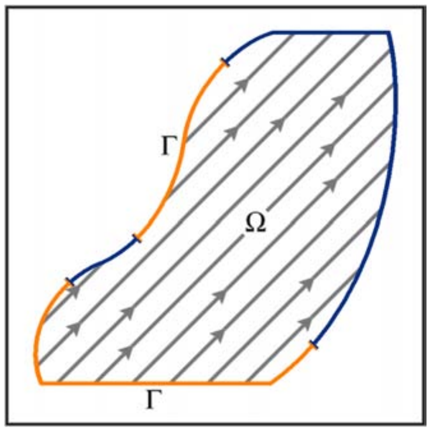
18.336 Numerical Methods for Partial Differential Equations

Fundamental Concepts

Domain $\Omega \subset \mathbb{R}^n$ with boundary $\partial\Omega$

$$\left\{ \begin{array}{lll} \text{PDE} & \text{in} & \Omega \\ \text{b.c.} & \text{on} & \Gamma \subset \partial\Omega \end{array} \right\}$$



PDE = "partial differential equation"
b.c. = "boundary conditions"
(if time involved, also i.c. = "initial conditions")

Def.: An expression of the form

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0, \quad x \in \Omega \subset \mathbb{R}^n \tag{1}$$

is called k^{th} order PDE,
where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times ... \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is given,
and $u : \Omega \to \mathbb{R}$ is the unknown.
A function u satisfying (1) is called *solution* of the PDE.

Du	$= (u_{x_1}, ..., u_{x_n})$	gradient (vector)
D^2u	$= \begin{pmatrix} u_{x_1x_1} & \cdots & u_{x_1x_n} \\ \vdots & \ddots & \vdots \\ u_{x_nx_1} & \cdots & u_{x_nx_n} \end{pmatrix}$	Hessian (matrix)
$\vdots$		$\vdots$ etc.

Def.: The PDE (1) is called...

(i) linear, if

$$\sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha u = f(x)$$

homogeneous, if $f = 0$

(ii) semilinear, if

$$\sum_{|\alpha|\leq k} a_\alpha(x) D^\alpha u + F_0(D^{k-1}u, ..., Du, u, x) = 0$$

(iii) quasilinear, if

$$\sum_{|\alpha|\leq k} a_\alpha(D^{k-1}u, ..., Du, u, x) \cdot D^\alpha u + F_0(D^{k-1}u, ..., Du, u, x) = 0$$

(iv) fully nonlinear, if neither (i), (ii) nor (iii).

Def.: An expression of the form

$$F(D^k u(x), D^{k-1}u(x), ..., Du(x), u(x), x) = 0, \quad x \in \Omega \subset \mathbb{R}^n$$

is called k^{th} order system of PDE,
where $F : \mathbb{R}^{mn^k} \times \mathbb{R}^{mn^{k-1}} \times ... \times \mathbb{R}^{mn} \times \mathbb{R}^m \times \Omega \to \mathbb{R}^m$
and $u : \Omega \to \mathbb{R}^m,\ u = (u^1, ..., u^m)$.

Typically: # equations = # unknowns , i.e. $n = m$.

Some examples:

$u_t + u_x = 0$ linear advection equation

$u_t = u_{xx}$ heat equation

$u_{xx} = f(x)$ Poisson equation (1D)

$\nabla^2 u = f$ Poisson equation (nD)

$u_t + cu_x = Du_{xx}$ convection diffusion equation

$u_t + (\frac{1}{2}u^2)_x = 0 \Leftrightarrow u_t + u\,u_x = 0$ Burgers' equation (quasilinear)

$\nabla^2 u = u^2$ a semilinear PDE

$u_{tt} = u_{xx}$ wave equation (1D)

$\begin{pmatrix} u \\ v \end{pmatrix}_t = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} u \\ v \end{pmatrix}_x$ wave equation, written as a system $u_{tt} = v_{xt} = v_{tx} = u_{xx}$

$u_t + uu_x = \epsilon u_{xxx}$ Korteweg-de-Vries equation

$\left\{ \begin{array}{l} \vec{u}_t + (\vec{u} \cdot \nabla)\vec{u} = -\nabla p + \nu \nabla^2 \vec{u} \\ \nabla \cdot \vec{u} = 0 \end{array} \right\}$ incompressible Navier-Stokes equation [dynamic-algebraic system]

$\left\{ \begin{array}{l} h_t + (uh)_x = 0 \\ u_t + uu_x = -gh_x \end{array} \right\}$ shallow water equations [system of hyberbolic conservation laws]

$|\nabla u| = 1$ Eikonal equation (nonlinear)

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