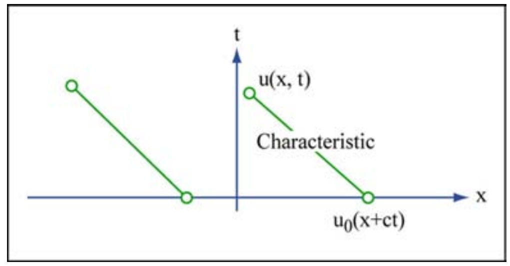
Finite Difference Methods for the One-Way Wave Equation

$$\begin{cases} u_t = cu_x \\ u(x,0) = u_0(x) \end{cases}$$

Solution: $u(x,t) = u_0(x+ct)$

Information travels to the left with velocity c.



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Three Approximations:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \begin{cases} c\dfrac{U_{j+1}^n - U_j^n}{\Delta x} & \text{upwind} \\ c\dfrac{U_j^n - U_{j-1}^n}{\Delta x} & \text{downwind} \\ c\dfrac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} & \text{centered} \end{cases}$$

Accuracy:

Taylor expansion of solution u

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} = u_t + \frac{1}{2}u_{tt}\Delta t + \frac{1}{6}u_{ttt}(\Delta t)^2 + O((\Delta t)^3)$$

$$\frac{u(x+\Delta x,t) - u(x,t)}{\Delta x} = u_x + \frac{1}{2}u_{xx}\Delta x + \frac{1}{6}u_{xxx}(\Delta x)^2 + O((\Delta x)^3)$$

$$\frac{u(x+\Delta x,t) - u(x-\Delta t,t)}{2\Delta x} = u_x + \frac{1}{6}u_{xxx}(\Delta x)^2 + O((\Delta x)^4)$$

Substitute into FD scheme:

Upwind: $\underbrace{u_t}_{=cu_x} + \underbrace{\frac{1}{2}u_{tt}\Delta t}_{=\frac{1}{2}c^2u_{xx}\Delta t} - cu_x - \frac{1}{2}cu_{xx}\Delta x + O(\Delta t^2) + O(\Delta x^2)$

Leading order error:

$\frac{1}{2}u_{tt}\Delta t - \frac{1}{2}cu_{xx}\Delta x = \frac{1}{2}c^2u_{xx}\Delta t - \frac{1}{2}cu_{xx}\Delta x = \frac{1}{2}cu_{xx}\Delta x(r-1)$

$= 0$ if $r = 1$

$$\boxed{r = \frac{c\Delta t}{\Delta x}}$$

Courant number

First order if $r \neq 1$

Downwind: Analogous: first order

Centered: $u_t + \frac{1}{2}u_{tt}\Delta t - cu_x - \frac{1}{6}cu_{xxx}\Delta x^2 + O(\Delta t^2) + O(\Delta x^4)$
$\Delta t \rightarrow$ First order in time
$\Delta x^2 \rightarrow$ Second order in space

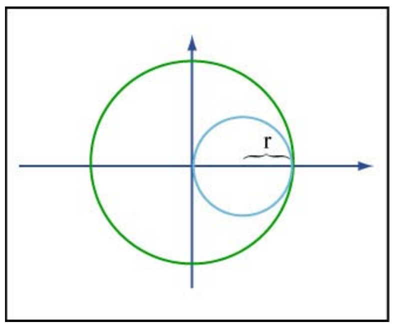
Stability:

Upwind: $\dfrac{G-1}{\Delta t} = c\dfrac{e^{ik\Delta x}-1}{\Delta x}$

$\Rightarrow G = 1 - r + re^{ik\Delta x}$

$\Rightarrow |G| \leq |1-r| + |re^{ik\Delta x}| = 1$, if $\underbrace{0 \leq r \leq 1}_{\text{CFL-Condition}}$

conditionally stable



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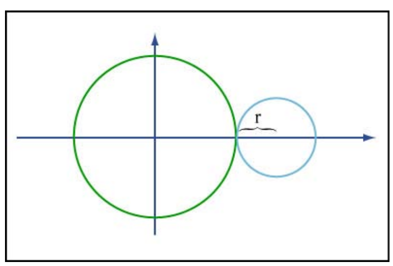
Downwind: $G = 1 + r - re^{-ik\Delta x}$

unstable



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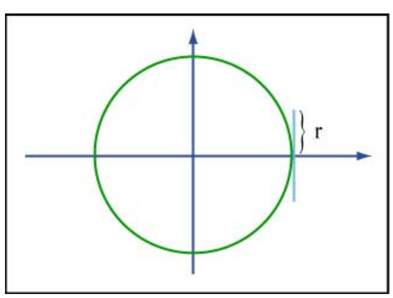
Centered: $G = 1 + r\dfrac{e^{ik\Delta x} - e^{-ik\Delta x}}{2} = 1 + ir\sin(k\Delta x)$

unstable



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Messages:

1. Upwind works (CFL-condition on stability)
2. Centered needs a fix

Add Diffusion:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \theta\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

Replace U_j^n by average:

$$\frac{U_j^{n+1} - \left(\frac{\lambda}{2}U_{j+1}^n + (1-\lambda)U_j^n + \frac{\lambda}{2}U_{j-1}^n\right)}{\Delta t} = c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x}$$

where $\lambda = 2\frac{\Delta t}{(\Delta x)^2}\theta$

How much diffusion?

Lax-Friedrichs:

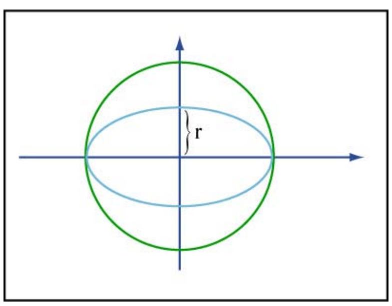
Eliminate U_j^n by $\lambda = 1 \Rightarrow \theta = \frac{(\Delta x)^2}{2\Delta t}$

$$U_j^{n+1} = \underbrace{\frac{1+r}{2}}_{\geq 0 (\text{for } |r|\leq 1)} U_{j+1}^n + \underbrace{\frac{1-r}{2}}_{\geq 0 (\text{for } |r|\leq 1)} U_{j-1}^n \qquad \boxed{r = \frac{c\Delta t}{\Delta x}}$$

Monotone scheme

Accuracy: First in time, Second in space (exercise)
Stability: $G = \underbrace{ir\sin(k\Delta x)}_{\text{central difference}} + \underbrace{\cos(k\Delta x)}_{\text{diffusion}}$

conditionally stable $|r| \leq 1$



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Lax-Wendroff:

Choose θ to get second order in time: $\theta = \frac{\Delta t}{2}c^2$

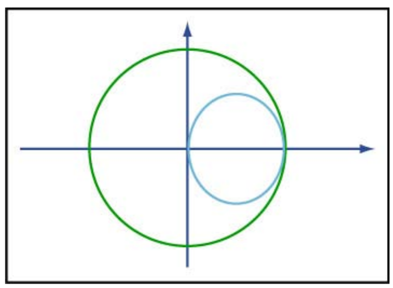
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \frac{\Delta t}{2}c^2\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

Accuracy: $u_t + \frac{1}{2}u_{tt}\Delta t + \frac{1}{6}u_{ttt}\Delta t^2 - cu_x - \frac{1}{6}cu_{xxx}\Delta x^2 - \frac{\Delta t}{2}c^2 u_{xx} - \frac{\Delta t}{24}c^2 u_{xxxx}\Delta x^2$
$= \frac{1}{6}u_{ttt}\Delta t^2 - \frac{1}{6}cu_{xxx}\Delta x^2 = O(\Delta t^2) + O(\Delta x^2)$
$u_t - cu_x = 0$
$\frac{1}{2}u_{tt}\Delta t - \frac{\Delta t}{2}c^2 u_{xx} = 0$

Stability: $\lambda = r^2$

$$\begin{aligned} G &= \frac{r^2 + r}{2}e^{ik\Delta x} + (1 - r^2) + \frac{r^2 - r}{2}e^{-ik(\Delta x)} \\ &= (1 - r^2) + r^2\cos(k\Delta x) + ir\sin(k\Delta x) \end{aligned}$$

Worst case: $k\Delta x = \pi \Rightarrow G = 1 - 2r^2$
Stable if $|r| \leq 1$



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