Modified Equation

Idea: Given FD approximation to PDE

Find another PDE which is approximated better by FD scheme.
Learn from new PDE about FD scheme.

Ex.: $u_t = cu_x$

Lax-Friedrichs: $\frac{U_j^{n+1}-U_j^n}{\Delta t} - c\frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} - \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{2\Delta t} = 0$

Taylor: $u_t + \frac{1}{2}u_{tt}\Delta t - cu_x - \frac{1}{6}cu_{xxx}\Delta x^2 - \frac{1}{2\Delta t}u_{xx}\Delta x^2 - \frac{1}{24\Delta t}u_{xxxx}\Delta x^4 + \dots$

$= (u_t - cu_x) + \frac{1}{2}\Big(\underbrace{u_{tt}}_{=c^2u_{xx}}\Delta t - u_{xx}\frac{\Delta x^2}{\Delta t}\Big) + \dots$

$= (u_t - cu_x) + \frac{1}{2}\Big(c^2\Delta t - \frac{\Delta x^2}{\Delta t}\Big)u_{xx}$

Modified equation:

$u_t - cu_x = \frac{\Delta x^2}{2\Delta t}(1-r^2)u_{xx}$ $\qquad$ $\boxed{r = \frac{c\Delta t}{\Delta x}}$

Advection-diffusion equation with diffusion constant

$D = \frac{\Delta x^2}{2\Delta t}(\underbrace{1}_{\text{added diffusion}} - \underbrace{r^2}_{\text{antidiffusion by central differencing}})$

Ex.: Upwind:

$u_t - cu_x = \frac{1}{2}c\Delta x(1-r)u_{xx}$ (exercise)

Compare:

For $c = 1, r = \frac{1}{2} \longrightarrow D_{\text{LF}} = \frac{3}{4}\Delta x$, $D_{\text{UW}} = \frac{1}{4}\Delta x$

Upwind less diffusive than LF.

Ex.: Lax-Wendroff

$u_t - cu_x = \frac{1}{6}c\Delta x^2(r^2-1)u_{xxx}$ (u_{xx} cancels by construction)

Advection-dispersion equation with dissipation constant

$\mu = -\frac{1}{6}c\Delta x^2(1-r^2)$

Disturbances behave like Airy's equation

Message:

First order methods behave diffusive.
Second order methods behave dispersive.

More on Advection Equation

$u_t + cu_x = 0$

So far:

1. Upwind:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \left\{ \begin{array}{ll} -c\dfrac{U_j^n - U_{j-1}^n}{\Delta x} & c > 0 \\ -c\dfrac{U_{j+1}^n - U_j^n}{\Delta x} & c < 0 \end{array} \right\} \rightarrow e = O(\Delta t) + O(\Delta x)$$

2. Lax-Friedrichs/Lax-Wendroff:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{U_{j+1}^n - U_{j-1}^n}{2\Delta x} + \theta \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

LF: $\theta = \dfrac{(\Delta x)^2}{2\Delta t} \rightarrow e = O(\Delta t) + O(\Delta x^2)$

LW: $\theta = \dfrac{\Delta t}{2} c^2 \;\; \rightarrow e = O(\Delta t^2) + O(\Delta x^2)$

Semidiscretization:

Central: $u_x = \dfrac{U_{j+1} - U_{j-1}}{2\Delta x} + O(\Delta x^2)$

$$\text{Matrix} \begin{bmatrix} (u_x)_1 \\ \vdots \\ (u_x)_k \end{bmatrix} = \frac{1}{2\Delta x} \underbrace{\begin{bmatrix} 0 & 1 & & -1 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 1 & & -1 & 0 \end{bmatrix}}_{=A} \cdot \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$$

$A^T = -A \Rightarrow$ eigenvalues purely imaginary

Need time discretization that is stable for $\dot{u} = \lambda u$ with $\lambda = i\mu, \;\; \mu \in \mathbb{R}$

Linear Stability for ODE:

Region of absolute stability $= \{\lambda \in \mathbb{C} : \text{method stable for } \dot{u} = \lambda u\}$

Ex.:

Forward Euler

$$u^{n+1} = u^n + \lambda \Delta t u^n = (1 + \lambda \Delta t) u^n$$

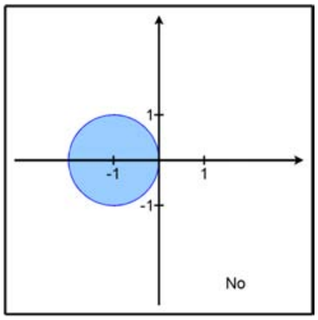
Stable if $|1 + \lambda \Delta t| \leq 1$

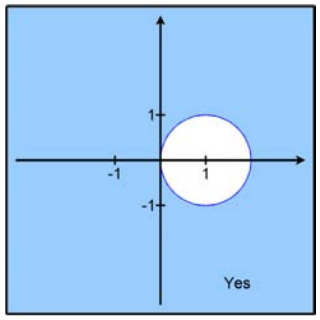
Backward Euler

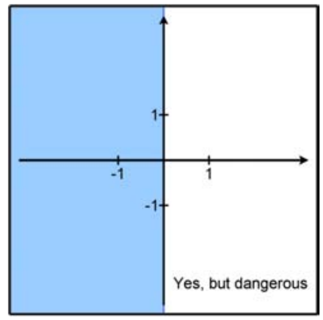
$$u^{n+1} = \frac{1}{1 - \lambda \Delta t} u^n$$

Trapezoidal

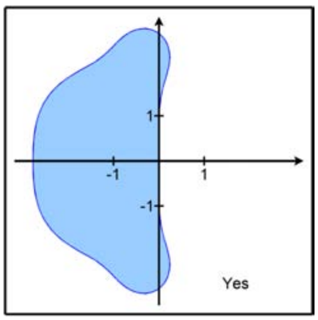
$$u^{n+1} = \frac{1 + \frac{1}{2}\lambda \Delta t}{1 - \frac{1}{2}\lambda \Delta t}$$



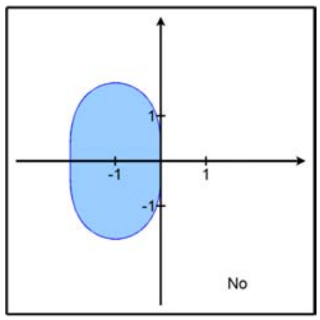




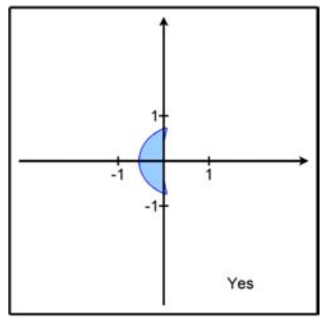
RK4



RK2



Adams-Bashforth 3

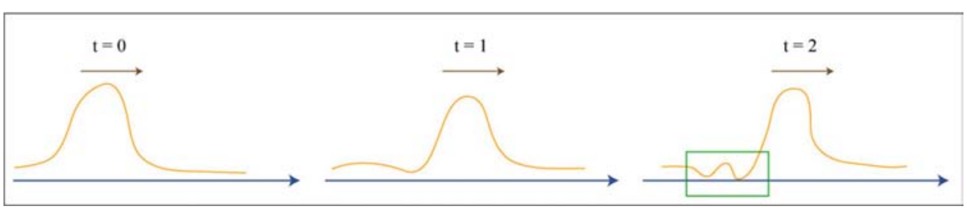


Can also use higher order discretization of u_x (up to spectral). If central $\Rightarrow$ need ODE solver for timestep that is stable for $\dot{u} = i\mu u$.

Spurious Oscillations

Stable does not imply "no oscillations."

Ex.: Lax-Wendroff



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Overshoots remain bounded $\Rightarrow$ stable. Still bad (e.g. density can become negative)

Total Variation:

$TV(u) = \sum_j |u_{j+1} - u_j| \approx \int |u_x(x)|dx$ "total up and down"

Method total variation diminishing (TVD), if
$TV(u^{n+1}) \leq TV(u^n)$.

Bad News: Any linear method for advection that is TVD,
is at most first order accurate.

[i.e.: high order $\to$ spurious oscillations]

Remedy: Nonlinear Methods:

1. Flux-/Slope- Limiters

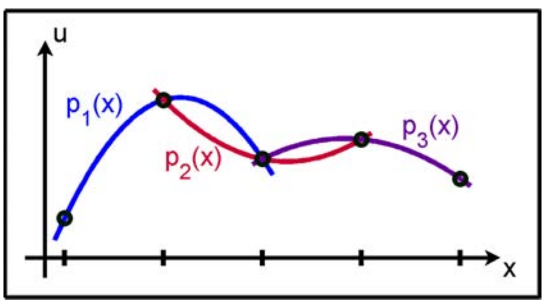
   $\rightsquigarrow$ conservation laws; limit flux $\to$ TVD

2. ENO/WENO

   (weighted) essentially non-oscillatory
   (essentially TVD; no noticeable spurious oscillations)

ENO/WENO

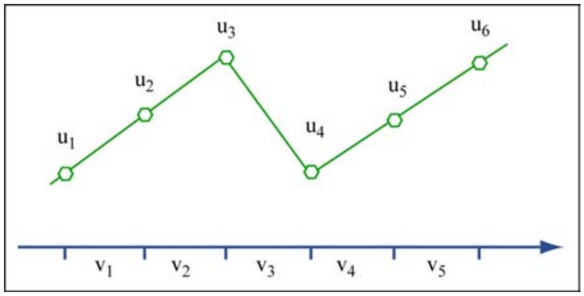
Approximate u_x by interpolation.



ENO: At each point consider multiple interpolating polynomials (through various choices of neighbors). Select the most "stable" one to define u_x.

WENO: Define u_x as weighted average of multiple interpolants.
Higher order when u smooth, no overshoots when u non-smooth.

Ex.: Fifth order WENO



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$$s_1 = \tfrac{13}{12}(v_1 - 2v_2 + v_3)^2 + \tfrac{1}{4}(v_1 - 4v_2 + 3v_3)^2$$
$$s_2 = \tfrac{13}{12}(v_2 - 2v_3 + v_4)^2 + \tfrac{1}{4}(v_2 - v_4)^2$$
$$s_3 = \tfrac{13}{12}(v_3 - 2v_4 + v_5)^2 + \tfrac{1}{4}(3v_3 - 4v_4 + v_5)^2 \qquad v_j = \frac{U_{j+1} - U_j}{\Delta x}$$
$$a_1 = \tfrac{1}{10}/(\epsilon + s_1)^2$$
$$a_2 = \tfrac{6}{10}/(\epsilon + s_2)^2 \qquad \epsilon = 10^{-6} \cdot \max_j ({v_j}^2)$$
$$a_3 = \tfrac{3}{10}/(\epsilon + s_3)^2$$
$$s_a = a_1 + a_2 + a_3$$
$$w_1 = \tfrac{a_1}{s_a}$$
$$w_2 = \tfrac{a_2}{s_a}$$
$$w_3 = \tfrac{a_3}{s_a}$$
$$w = \tfrac{1}{6}(w_1 \cdot (2v_1 - 7v_2 + 11v_3) + w_2 \cdot (-v_2 + 5v_3 + 2v_4) + w_3 \cdot (2v_3 + 5v_4 - v_5))$$
$$\left\{ \begin{array}{l} \text{Left sided approximation to } u_x \text{ at } x_4 \\ \text{Right sided approximation to } u_x \text{ at } x_3 \end{array} \right\}$$

$u_t + cu_x = 0$

Upwind WENO with FE:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \left\{ \begin{array}{ll} -c \cdot \text{WENO}_{\text{left}} & U_j^n > 0 \\ -c \cdot \text{WENO}_{\text{right}} & U_j^n \leq 0 \end{array} \right\}$$

TVD time stepping

Consider method that is TVD with FE.
Is it also TVD with high order time stepping?
In general: "no."
But for special class of ODE schemes: "yes."
Strong Stability Preserving (SSP) methods

Ex.: FE $(u^n) = u^n + \Delta t f(u^n)$

RK3-TVD

$u^{n+1} = \frac{1}{3}u^n + \frac{2}{3}$ FE $(\frac{3}{4}u^n + \frac{1}{4}\text{FE}(\text{FE}(u^n)))$

Convex combination of FE steps

$\Rightarrow$ Preserves TVD property

Compare: Classical RK4 cannot by written this way.

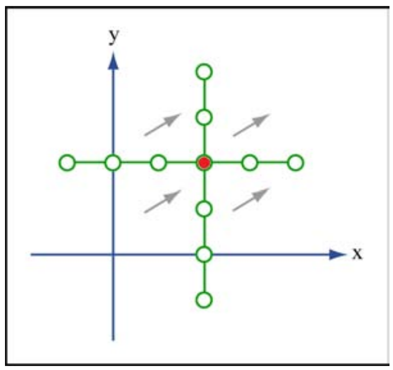
It is not SSP.

Popular approach for linear advection:

$u_t + cu_x = 0$

RK3-TVD in time, upwinded WENO5 in space.

2D/3D: Tensor product in space.



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WENO5-stencil

18.336 Numerical Methods for Partial Differential Equations
Spring 2009