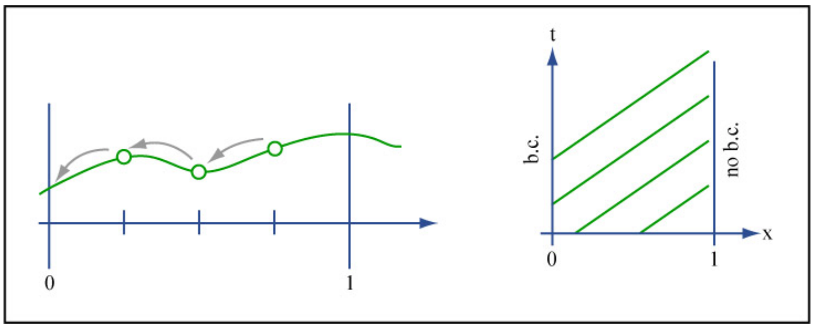
Intermezzo:
Boundary Conditions for Advection

Linear advection

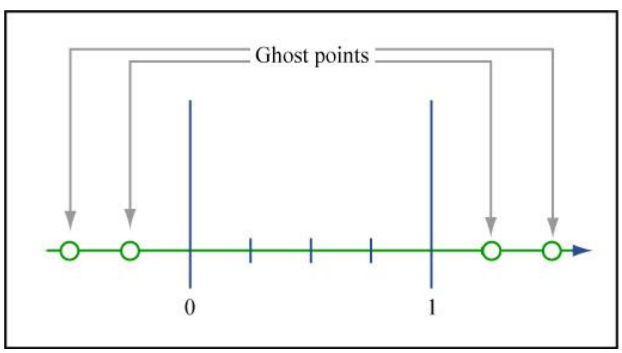
$$\begin{cases} u_t + u_x = 0 \\ x \in [0,1] \end{cases}$$



Upwind (first order) treats boundary conditions naturally correctly.

LF, LW: Need artificial/numerical boundary conditions at $x = 1$, that acts as close to "do nothing" as possible.

WENO: Need information outside of domain.



Which values?

Extrapolate solution; works if u smooth.
E.g., constant/linear/etc. extrapolation.

Numerical Methods for Conservation Laws

Naive Schemes

Ex.: Burgers'

$$u_t + uu_x = 0 \qquad \text{with } u > 0$$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = -U_j^n \overbrace{\frac{U_j^n - U_{j-1}^n}{\Delta x}}^{\text{upwind}}$$

Non-conservative; fine if $u \in C^1$.

But: Wrong if u has shocks (see pset 5).

Conservative FD Methods

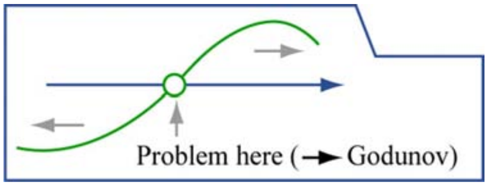
$$u_t + (f(x))_x = 0$$

Discretize in conservation form

$\Rightarrow$ conservation $\Rightarrow$ correct shock speeds

Upwind:

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \left\{ \begin{array}{ll} -\frac{f(U_j^n) - f(U_{j-1}^n)}{\Delta x} & \text{if } f'(U_j^n) \geq 0 \\ -\frac{f(U_{j+1}^n) - f(U_j^n)}{\Delta x} & \phantom{\text{if }} f'(U_j^n) < 0 \end{array} \right\}$$



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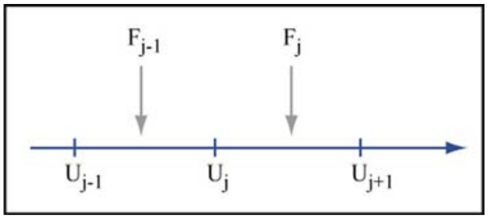
Lax-Friedrichs:

$$\frac{U_j^{n+1} - \frac{1}{2}(U_{j+1}^n + U_{j-1}^n)}{\Delta t} + \frac{f(U_{j+1}^n) - f(U_{j-1}^n)}{2\Delta x} = 0$$

(no straightforward LW, since based on linear Taylor expansion)

Numerical Flux Function

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{{F_j}^n - {F_{j-1}}^n}{\Delta x} = 0$$



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$$\text{Upwind: } {F_j}^n = \left\{ \begin{array}{lll} f(U_j^n) & \text{if } \frac{f(U_{j+1}^n) - f(U_j^n)}{U_{j+1}^n - U_j^n} & \geq 0 \\ f({U_{j+1}}^n) & & < 0 \end{array} \right\}$$

$$\text{LF : } {F_j}^n = \tfrac{1}{2}(f(U_j^n) + f(U_{j+1}^n)) - \frac{\Delta x}{2\Delta t}(U_{j+1}^n - U_j^n)$$