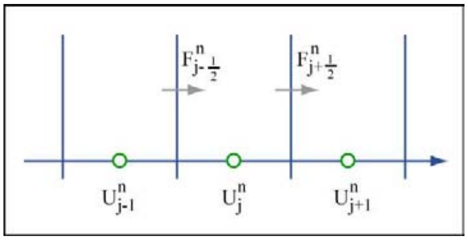
Finite Volume Methods (FVM)

FD: $U_j^n \approx$ function value $u(j\Delta x, n\Delta t)$

FV: $U_j^n \approx$ cell average $\frac{1}{\Delta x}\int_{(j-\frac{1}{2})\Delta x}^{(j+\frac{1}{2})\Delta x} u(x, n\Delta t)dx$

Fluxes through cell boundaries

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{F_{j+\frac{1}{2}}{}^n - F_{j-\frac{1}{2}}{}^n}{\Delta x} = 0$$



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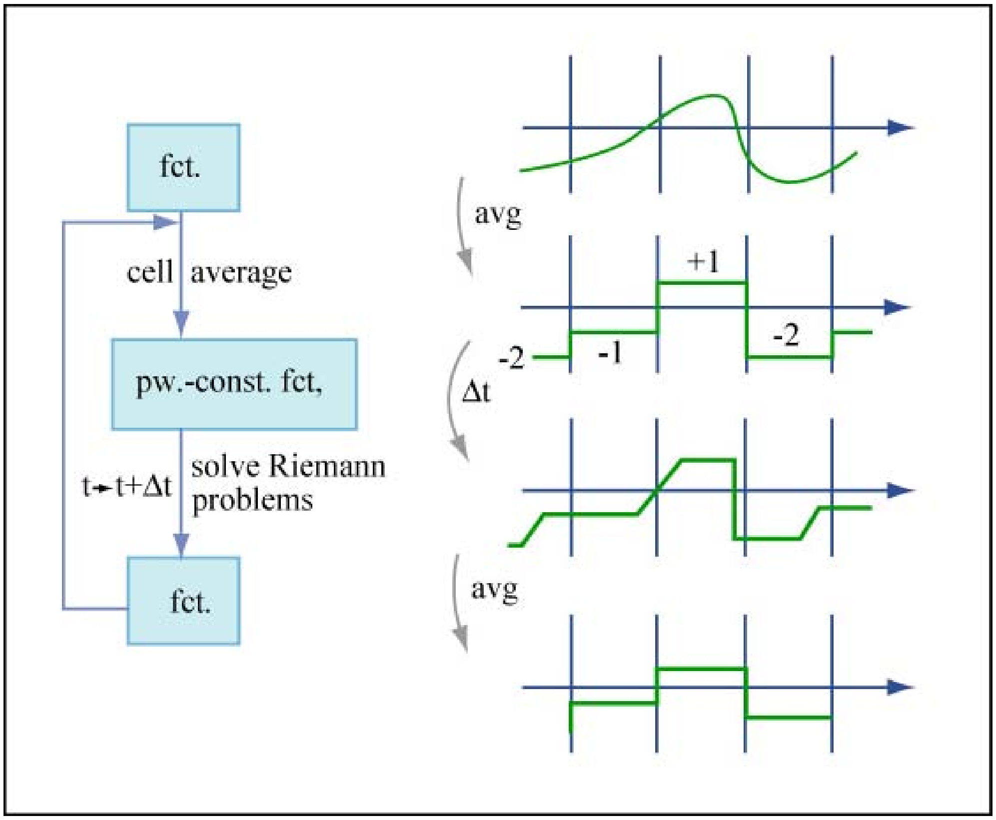
Godunov Method

REA = Reconstruct-Evolve-Average

Burgers' equation



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CFL Condition: $\Delta t \leq C \cdot \Delta x$

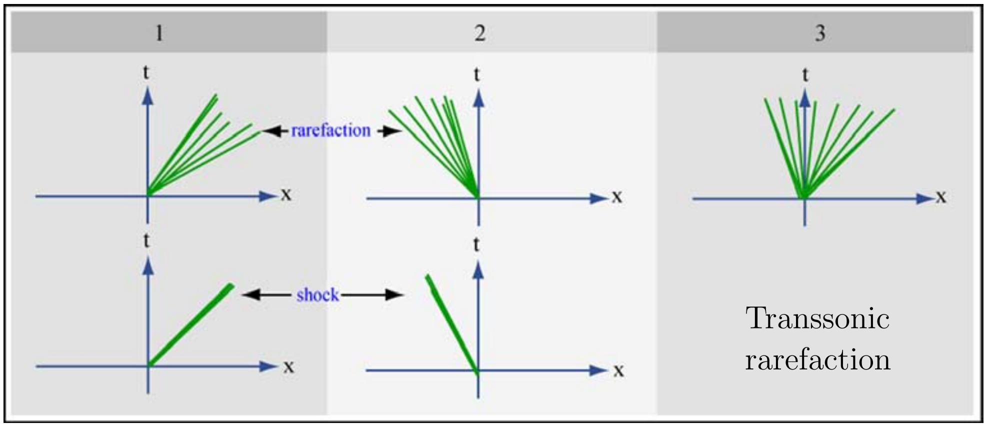
Local RP do not interact

If $f''(u) > 0$ (convex flux function)

$$F^n_{j-\frac{1}{2}} = \left\{ \begin{array}{lll} f(U^n_{j-1}) & & U^n_{j-1} > u_s,\ s > 0 \\ f(U^n_j) & \text{if} & U^n_j < u_s,\ s < 0 \\ f(U_s) & & U^n_{j-1} < u_s < U^n_j \end{array} \right\} \quad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

$$s = \frac{f(u^n_j) - f(u^n_{j-1})}{u^n_j - u^n_{j-1}} \qquad \text{Shock speed}$$

$$f'(u_s) = 0 \qquad \text{Sonic point} \quad [\text{Burgers': } u_s = 0\ ]$$



If no transsonics occur, we recover exactly upwind. Image by MIT OpenCourseWare.

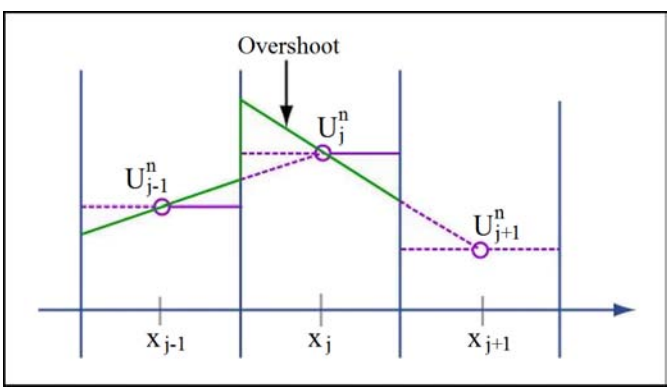
$\Rightarrow$ FV $\overset{\text{Close Relation}}{\longleftrightarrow}$ FD.

High Order Methods

Linear case: Taylor series approach $\rightarrow$ LW

FD: Larger stencils

FV: Reconstruct with linear, quadratic, etc. functions in each cell.



$\sigma_j{}^n = 0$

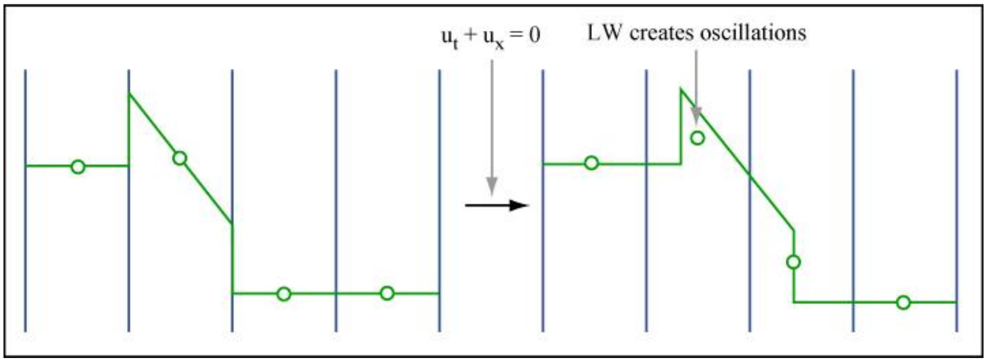
$\Rightarrow$ Godunov's method

$$\sigma_j{}^n = \frac{u^n_{j+1} - u^n_j}{\Delta x}$$

$\Rightarrow$ Lax-Wendroff

$\tilde{u}^n(x,t_n) = U^n_j + \sigma^n_i \cdot (x - x_j)$ Image by MIT OpenCourseWare.

Riemann Problem



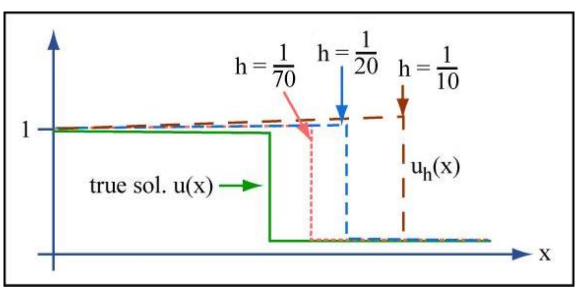
High order not TVD $\Rightarrow$ need limiters.

Image by MIT OpenCourseWare.

Nonlinear Stability

Property	Conservation law	Numerical scheme
Monotone $\Downarrow$	Initial conditions $v_0(x) \geq u_0(x)\ \forall x$ $\Rightarrow v(x,t) \geq u(x,t)\ \forall x, t.$	$V_j^n \geq U_j^n\ \forall j$ $\Rightarrow V_j^{n+1} \geq U_j^{n+1}\ \forall j$
L^1-contracting $\Downarrow$	$\lVert u(\cdot,t_2)\rVert_{L^1} \leq \lVert u(\cdot,t_1)\rVert_{L^1}$ $\forall t_2 \geq t_1$	$\lVert U^{n+1} - V^{n+1}\rVert_1 \leq \lVert U^n - V^n\rVert_1$ $[\lVert U\rVert_1 = \Delta x \sum_j \lvert u_j\rvert]$
TVD $\Downarrow$	$\mathrm{TV}(u(\cdot,t_2)) \leq \mathrm{TV}(u(\cdot,t_1))$ $\forall t_2 \geq t_1$ $[\mathrm{TV}(u) = \int \lvert u(x)\rvert dx]$	$\mathrm{TV}(U^{n+1}) \leq \mathrm{TV}(U^n)$ $[\mathrm{TV}(U) = \sum_j \lvert U_{j+1} - U_j\rvert]$
Monotonicity preserving	$u_x(\cdot,t_1) \geq 0 \Rightarrow u_x(\cdot,t_2) \geq 0$ if $t_2 > t_1$	$U_j^n \geq U_{j+1}^n\ \forall j$ $\Rightarrow U_j^{n+1} \geq U_{j+1}^{n+1}\ \forall j$
TVB (bounded)		$\mathrm{TV}(U^{n+1}) \leq (1+\alpha\Delta t)\cdot$ TV (U^n) α independent of Δt

Remark: Discontinuous solution $\Rightarrow L^1-$norm is appropriate



$$\lVert u_h(x) - u(x)\rVert_{L^\infty} = 1\ \forall h$$

$$\lVert u_h(x) - u(x)\rVert_{L^1} \xrightarrow{h\to 0} 0$$

Image by MIT OpenCourseWare.

Theorem (Godunov):

A linear, monoticity preserving method is at most first order accurate.
$\Rightarrow$ Need nonlinear schemes.

High Resolution Methods

A. Flux Limiters

$u_t + (f(u))_x = 0$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} + \frac{F_j^n - F_{j-1}^n}{\Delta x} = 0$$

Use two fluxes:

- TVD-flux (e.g. upwind) $\hat{F}$
- High order flux $\tilde{F}$

Smoothless indicator:

$$\theta_j = \frac{u_j - u_{j-1}}{u_{j+1} - u_j} \begin{cases} \approx 1 & \text{where smooth} \\ \text{away from } 1 & \text{near shocks} \end{cases}$$

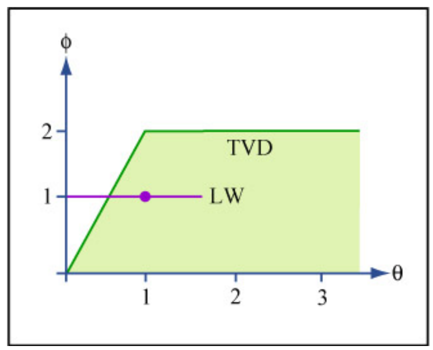
Flux: $F_j = \hat{F}_j + (\tilde{F}_j - \hat{F}_j) \cdot \Phi(\theta_j)$

Ex.: $u_t + cu_x = 0$

$$F_j = \underbrace{cU_j}_{F_{\text{upwind}}} + \underbrace{\tfrac{c}{2}\left(1 - c\tfrac{\Delta t}{\Delta x}\right) \cdot (U_{j+1} - U_j) \cdot \Phi(\theta_j)}_{F_{\text{LW}} - F_{\text{upwind}}}$$

Conditions for $\Phi(\theta)$:

- TVD: $0 \leq \Phi(\theta) \leq 2\theta$
  $0 \leq \Phi(\theta) \leq 2$
- Second order: $\Phi(1) = 1$
  Φ continuous



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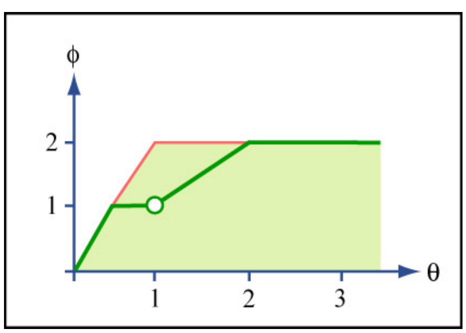
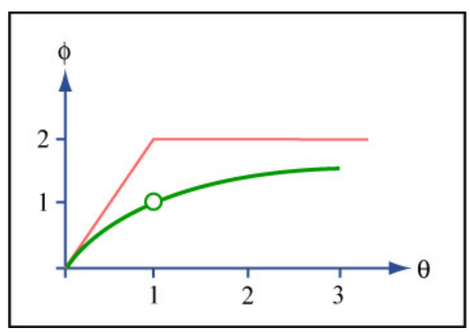
Two Popular Limiters:

Superbee



$\Phi(\theta) = \max(0, \min(1, 2\theta), \min(\theta, 2))$

van Leer



$$\Phi(\theta) = \frac{|\theta| + \theta}{1 + |\theta|}$$

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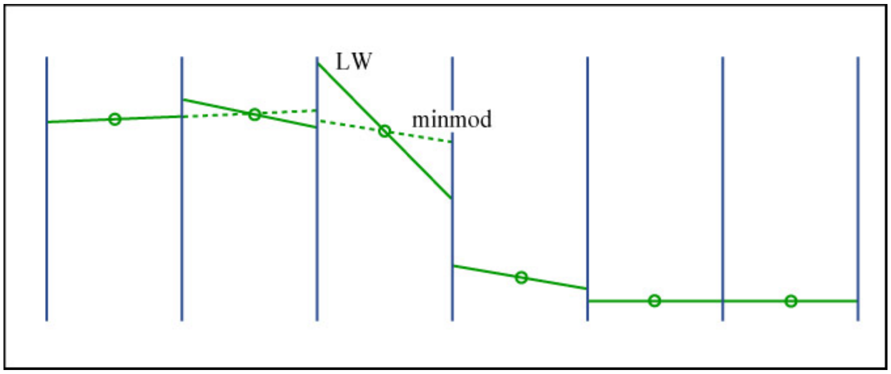
B. Slope Limiters



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Upwind: $\sigma_j = 0$

LW: $\sigma_j = \dfrac{U_{j+1} - U_j}{\Delta x}$

Minmod-limiter:

$$\sigma_j = \text{minmod}\left(\frac{U_j - U_{j-1}}{\Delta x}, \frac{U_{j+1} - U_j}{\Delta x}\right)$$

$$\text{minmod}(a, b) = \left\{ \begin{array}{lllll} a & & |a| < |b| & \& & ab > 0 \\ b & \text{if} & |a| > |b| & \& & ab > 0 \\ 0 & & & & ab < 0 \end{array} \right\}$$

Many more...

Slope limiters $\overset{\text{relation}}{\longleftrightarrow}$ Flux limiters