Well-Posedness

Def.: A PDE is called well-posed (in the sense of Hadamard), if

(1) a solution exists
(2) the solution is unique
(3) the solution depends continuously on the data
(initial conditions, boundary conditions, right hand side)

Careful: Existence and uniqueness involves boundary conditions

Ex.: $u_{xx} + u = 0$

a) $u(0) = 0, u(\frac{\pi}{2}) = 1 \Rightarrow$ unique solution $u(x) = \sin(x)$
b) $u(0) = 0, u(\pi) = 1 \Rightarrow$ no solution
c) $u(0) = 0, u(\pi) = 0 \Rightarrow$ infinitely many solutions: $u(x) = A\sin(x)$

Continuous dependence depends on considered metric/norm.
We typically consider $||\cdot||_{L^\infty}, ||\cdot||_{L^2}, ||\cdot||_{L^1}$.

Ex.:

$$\left\{\begin{array}{ll} u_t = u_{xx} & \text{heat equation} \\ u(0,t) = u(1,t) = 0 & \text{boundary conditions} \\ u(x,0) = u_0(x) & \text{initial conditions} \end{array}\right\} \text{well-posed}$$

$$\left\{\begin{array}{ll} u_t = -u_{xx} & \text{backwards heat equation} \\ u(0,t) = u(1,t) & \text{boundary conditions} \\ u(x,0) = u_0(x) & \text{initial conditions} \end{array}\right\} \begin{array}{l}\text{no continuous dependence} \\ \text{on initial data [later]}\end{array}$$

Notions of Solutions

Classical solution

k^{th} order PDE $\Rightarrow u \in C^k$

Ex.: $\nabla^2 u = 0 \Rightarrow u \in C^\infty$

$$\left\{\begin{array}{l} u_t + u_x = 0 \\ u(x,0) \in C^1 \end{array}\right\} \Rightarrow u(x,t) \in C^1$$

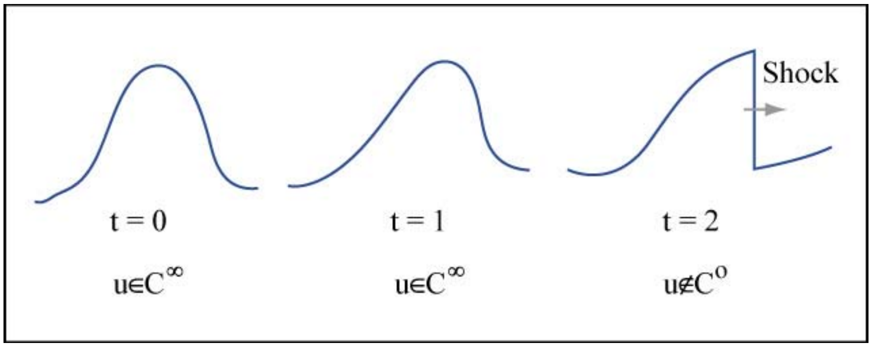
Weak solution
k^{th} order PDE, but $u \notin C^k$.

Ex.: Discontinuous coefficients

$$\left\{\begin{array}{l}(b(x)u_x)_x = 0 \\ u(0) = 0 \\ u(1) = 1 \\ b(x) = \left\{\begin{array}{ll} 1 & x < \frac{1}{2} \\ 2 & x \geq \frac{1}{2}\end{array}\right\}\end{array}\right\} \Rightarrow u(x) = \left\{\begin{array}{ll}\frac{4}{3}x & x < \frac{1}{2} \\ \frac{2}{3}x + \frac{1}{3} & x \geq \frac{1}{2}\end{array}\right.$$

Ex.: Conservation laws

$u_t + (\frac{1}{2}u^2)_x = 0$ Burgers' equation



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Fourier Methods for Linear IVP

IVP = initial value problem

$u_t = u_x$	advection equation
$u_t = u_{xx}$	heat equation
$u_t = u_{xxx}$	Airy's equation
$u_t = u_{xxxx}$	

a) on whole real axis: $u(x,t) = \int_{w=-\infty}^{w=+\infty} e^{iwx}\hat{u}(w,t)dw$ Fourier transform

b) periodic case $x \in [-\pi, \pi[$: $u(x,t) = \sum_{k=-\infty}^{+\infty} \hat{u}_k(t)e^{ikx}$ Fourier series (FS)

Here case b).

PDE: $\frac{\partial u}{\partial t}(x,t) - \frac{\partial^n u}{\partial x^n}(x,t) = 0$

insert FS: $\sum_{k=-\infty}^{+\infty}\left(\frac{d\hat{u}_k}{dt}(t) - (ik)^n\hat{u}_k(t)\right)e^{ikx} = 0$

Since $(e^{ikx})_{k\in\mathbb{Z}}$ linearly independent:

$\frac{d\hat{u}_k}{dt} = (ik)^n\hat{u}_k(t)$ ODE for each Fourier coefficient

Solution: $\hat{u}_k(t) = e^{(ik)^n t} \underbrace{\hat{u}_k(0)}_{\text{Fourier coefficient of initial conditions: } \hat{u}_k(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi} u_0(x)e^{-ikx}dx}$

$$\Rightarrow \boxed{u(x,t) = \sum_{k=-\infty}^{+\infty} \hat{u}_k(0) e^{ikx} e^{(ik)^n t}}$$

$n=1$:	$u(x,t) = \sum_k \hat{u}_k(0) e^{ik(x+t)}$	all waves travel to left with velocity 1
$n=2$:	$u(x,t) = \sum_k \hat{u}_k(0) e^{ikx} e^{-k^2 t}$	frequency k decays with $e^{-k^2 t}$
$n=3$:	$u(x,t) = \sum_k \hat{u}_k(0) e^{ik(x-k^2 t)}$	frequency k travels to right with velocity $k^2 \to$ dispersion
$n=4$:	$u(x,t) = \sum_k \hat{u}_k(0) e^{ikx} e^{k^4 t}$	all frequencies are amplified $\to$ unstable

Message:

For linear PDE IVP, study behavior of waves e^{ikx}.

The ansatz $u(x,t) = e^{-iwt} e^{ikx}$ yields a dispersion relation of w to k.

The wave e^{ikx} is transformed by the growth factor $e^{-iw(k)t}$.

Ex.:

wave equation:	$u_{tt} = c^2 u_{xx}$	$w = \pm ck$	conservative	$\lvert e^{\pm ickt}\rvert = 1$
heat equation:	$u_t = du_{xx}$	$w = -idk^2$	dissipative	$\lvert e^{-dk^2 t}\rvert \to 0$
conv.-diffusion:	$u_t = cu_x + du_{xx}$	$w = -ck - idk^2$	dissipative	$\lvert e^{ickt} e^{-dk^2 t}\rvert \to 0$
Schrödinger:	$iu_t = u_{xx}$	$w = -k^2$	dispersive	$\lvert e^{ik^2 t}\rvert = 1$
Airy equation:	$u_t = u_{xxx}$	$w = k^3$	dispersive	$\lvert e^{-ik^3 t}\rvert = 1$