Systems of IVP

Solution has multiple components: $\vec{u}(x,t) = \begin{bmatrix} u_1(x,t) \\ \vdots \\ u_m(x,t) \end{bmatrix}$

m equations

<u>Uncoupled</u> (trivial)

$u_t = u_{xx}$
$v_t = v_{xx}$
Solve independently

<u>Triangular</u> (easy)

(1) $u_t + uu_x = 0$ velocity field
(2) $\rho_t + u\rho_x = d\rho_{xx}$ density of pollutant
Solve first (1), then (2)

<u>Fully Coupled</u> (hard)

$$\left\{ \begin{array}{l} h_t + (uh)_x = 0 \\ u_t + uu_x + gh_x = 0 \end{array} \right\} \quad \text{shallow water equations}$$

$$\Leftrightarrow \begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} uh \\ \frac{1}{2}u^2 + gh \end{bmatrix}_x = 0$$

hyperbolic conservation law

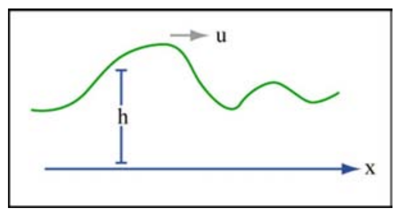

Image by MIT OpenCourseWare.

<u>Linear Hyperbolic Systems</u>

Linearize SW equations around base flow $\bar{h}, \bar{u}$:

$$\begin{bmatrix} h \\ u \end{bmatrix}_t + \begin{bmatrix} \bar{u} & \bar{h} \\ g & \bar{u} \end{bmatrix} \cdot \begin{bmatrix} h \\ u \end{bmatrix}_x = 0$$

Linear system:

$(*)\ \vec{u}_t + A \cdot \vec{u}_x = 0 \qquad A \in \mathbb{R}^{m\times m}$

SW: $A = \begin{bmatrix} \bar{u} & \bar{h} \\ g & \bar{u} \end{bmatrix} \qquad \lambda = \bar{u} \pm \sqrt{g\bar{h}}$

$(*)$ is called hyperbolic, if A is diagonalizable with real eigenvalues, and strictly hyperbolic, if the eigenvalues are distinct.

$A = R \cdot D \cdot R^{-1}$; change of coordinates: $\vec{v} = R^{-1} \cdot \vec{u}$
$\Rightarrow \vec{v}_t = D \cdot \vec{v}_x = 0$ Decoupled system
$(v_p)_t + \lambda_p (v_p)_x = 0\ \forall p = 1, \ldots, m$
$\Rightarrow v_p(x,t) = v_p(x - \lambda_p t, 0)$ Simple wave
Solution is superposition of simple waves

<u>Numerics</u>: Implement simple waves into Godunov's method.

Wave Equation

$\boxed{\text{1D}}$ $u_{tt} = c^2 u_{xx}$

$$\Leftrightarrow \partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix}$$

<u>Ex.</u>: Maxwell's Equations

$$\begin{cases} E_t = cH_x \\ H_t = cE_x \end{cases}$$

$$\Rightarrow \begin{cases} E_{tt} = (E_t)_t = (cH_x)_t = c(H_t)_x = c(cE_x)_x = c^2 E_{xx} \\ H_{tt} = \cdots = c^2 H_{xx} \end{cases}$$

<u>Schemes based on hyperbolic systems</u>

$$\begin{cases} \varphi = u + v \\ \psi = u - v \end{cases} \to \partial_t \begin{pmatrix} \varphi \\ \psi \end{pmatrix} = \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} \cdot \partial_x \begin{pmatrix} \varphi \\ \psi \end{pmatrix} \to \begin{cases} \varphi_t - c\varphi_x = 0 \\ \psi_t + c\psi_x = 0 \end{cases}$$

<u>Upwind for φ, ψ</u>:

$$\begin{cases} \dfrac{{\varphi_j}^{n+1} - {\varphi_j}^n}{\Delta t} = c\dfrac{{\varphi_{j+1}}^n - {\varphi_j}^n}{\Delta x} \\ \dfrac{{\psi_j}^{n+1} - {\psi_j}^n}{\Delta t} = -c\dfrac{{\psi_j}^n - {\psi_{j-1}}^n}{\Delta x} \end{cases}$$

$u = \frac{1}{2}(\varphi + \psi)$ and $v = \frac{1}{2}(\varphi - \psi)$

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \frac{1}{2}\left(\frac{{\varphi_j}^{n+1} - {\varphi_j}^n}{\Delta t} + \frac{{\psi_j}^{n+1} - {\psi_j}^n}{\Delta t}\right) = \frac{c}{2}\left(\frac{{\varphi_{j+1}}^n - {\varphi_j}^n}{\Delta x} + \frac{{\psi_j}^n - {\psi_{j-1}}^n}{\Delta x}\right)$$

$$= \frac{c}{2}\left(\frac{U_{j+1}^n - U_j^n}{\Delta x} + \frac{{V_{j+1}}^n - {V_j}^n}{\Delta x} - \frac{U_j^n - U_{j-1}^n}{\Delta x} + \frac{{V_j}^n - {V_{j-1}}^n}{\Delta x}\right)$$

$$= c\frac{{V_{j+1}}^n - {V_{j-1}}^n}{2\Delta x} + \frac{c\Delta x}{2} \cdot \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

$$\frac{{V_j}^{n+1} - {V_j}^n}{\Delta t} = \cdots = c\frac{U_{j+1}^n - U_{j-1}^n}{\Delta x} + \frac{c\Delta x}{2} \cdot \frac{{V_{j+1}}^n - 2{V_j}^n + {V_{j-1}}^n}{\Delta x^2}$$

<u>Lax-Friedrichs-like Scheme for u, v</u>:

$$\tfrac{1}{\Delta t}\left(\begin{pmatrix} U \\ V \end{pmatrix}_j^{n+1} - \begin{pmatrix} U \\ V \end{pmatrix}_j^n\right) = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \cdot \tfrac{1}{2\Delta x}\left(\begin{pmatrix} U \\ V \end{pmatrix}_{j+1}^n - \begin{pmatrix} U \\ V \end{pmatrix}_{j-1}^n\right) + \underbrace{\frac{c\Delta x}{2}\begin{pmatrix} U_{xx} \\ V_{yy} \end{pmatrix}}_{\text{artificial diffusion}}$$

Stable (check by von-Neumann stability analysis).

<u>More accurate schemes</u>: Use WENO and SSP-RK for φ, ψ.

Leapfrog Method

$$u_{tt} = c^2 u_{xx}$$

$$\rightarrow \frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{\Delta t^2} = c^2 \, \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

(Two-step method)

Accuracy:

$$u_{tt} + \tfrac{1}{2} u_{tttt} \Delta t^2 - c^2 u_{xx} - \tfrac{1}{12} c^2 u_{xxxx} \Delta x^2 = O(\Delta t^2) + O(\Delta x^2)$$

(using $u_{tt} - c^2 u_{xx} = 0$) second order

Stability:

$$\frac{G^2 - 2G + 1}{\Delta t^2} = c^2 G \frac{e^{ik\Delta x} - 2 + e^{-ik\Delta x}}{\Delta x^2}$$

$$\Rightarrow G^2 - 2G + 1 = 2r^2(\cos(k\Delta x) - 1) \cdot G$$

$r = \dfrac{c\Delta t}{\Delta x}$ Courant number

$$\Rightarrow G - 2\underbrace{\left(1 - r^2(1 - \cos(k\Delta x))\right)}_{=a} \cdot G + 1 = 0$$

$$\Rightarrow G = a \pm \sqrt{a^2 - 1}$$

If $|a| > 1 \Rightarrow$ one solution with $|G| > 1 \Rightarrow$ unstable
If $|a| \leq 1 \Rightarrow G = a \pm i\sqrt{1 - a^2} \Rightarrow |G|^2 = a^2 + (1 - a^2) = 1 \Rightarrow$ stable

Have $1 - \cos(k\Delta x) \in [0, 2]$, thus: $|a| \leq 1 \Leftrightarrow |r| \leq 1$
Leapfrog conditionally stable for $|r| \leq 1$.

Staggered Grids

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ v \end{pmatrix}$$

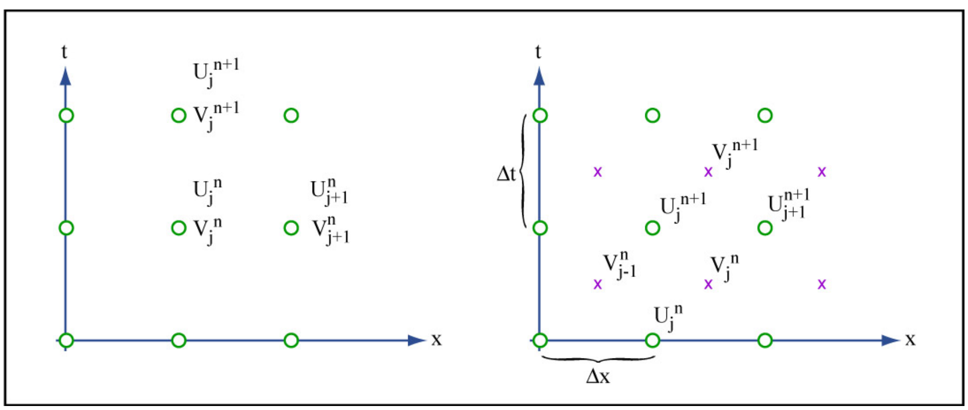

Image by MIT OpenCourseWare.

Collocation Grid

Central differencing requires artificial diffusion

Staggered Grid

Central differencing comes naturally

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} = c\dfrac{{V_j}^n - {V_{j-1}}^n}{\Delta x} \\ \dfrac{{V_j}^{n+1} - {V_j}^n}{\Delta t} = c\dfrac{U_{j+1}^{n+1} - U_j^{n+1}}{\Delta x} \end{cases}$$

Both are explicit central differences.

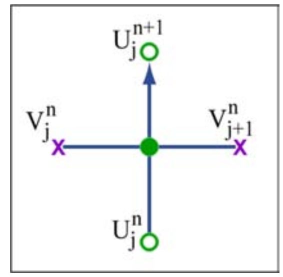

Image by MIT OpenCourseWare.

Two step update:

$$\frac{U_j^{n+1} - 2U_j^n + U_j^{n+1}}{\Delta t^2} = \frac{1}{\Delta t}\left(\frac{U_j^{n+1} - U_j^n}{\Delta t} - \frac{U_j^n - U_j^{n-1}}{\Delta t}\right)$$

$$= \frac{c}{\Delta t}\left(\frac{{V_j}^n - {V_{j-1}}^n}{\Delta x} - \frac{{V_j}^{n-1} - {V_{j-1}}^{n-1}}{\Delta x}\right) = \frac{c}{\Delta x}\left(\frac{{V_j}^n - {V_j}^{n-1}}{\Delta t} - \frac{{V_{j-1}}^n - {V_{j-1}}^{n-1}}{\Delta t}\right)$$

$$= \frac{c^2}{\Delta x}\left(\frac{U_{j+1}^n - U_j^n}{\Delta x} - \frac{U_j^n - U_{j-1}^n}{\Delta x}\right) = c^2\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2}$$

Equivalent to Leapfrog.