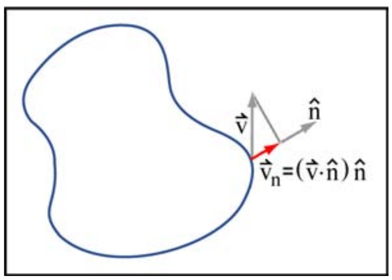
Level Set Method

$\left\{\begin{array}{ll} 2D: & \text{Moving Curve} \\ 3D: & \text{Moving Surface} \end{array}\right\}$ Orientable, with inside and outside region

Ex.:
- Interface between water and oil (surface tension)
- Propagating front of bush fire
- Deformable elastic solid

Movement of surface under velocity field $\vec{v}$.
Tangential motion does not change surface.
Only velocity component normal to surface is important.



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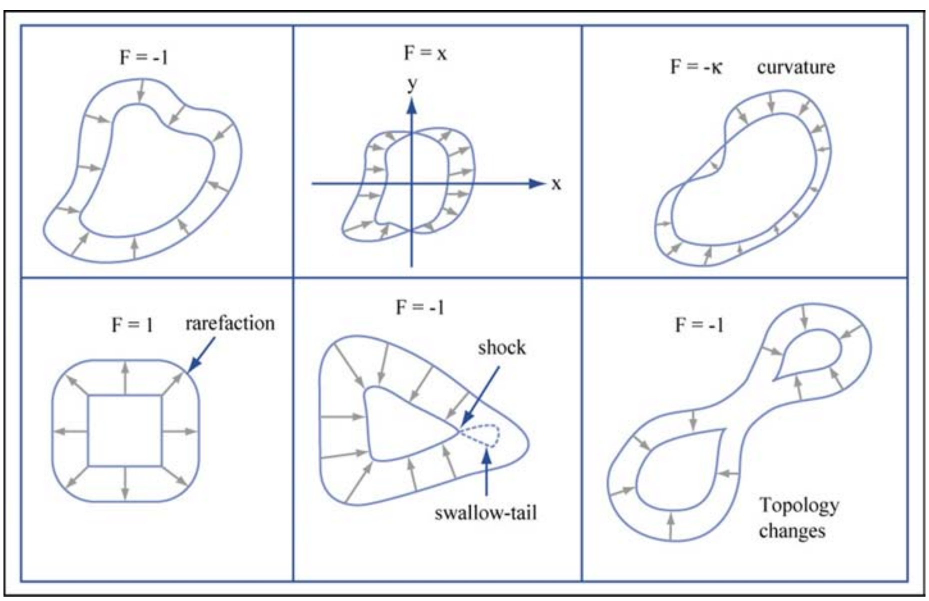
Effective velocity field: $\boxed{\vec{v} = F\hat{n}}$

Ex.:



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Explicit Tracking

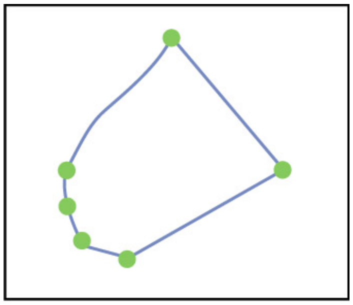
I. Lagrangian Markers:

- Place markers on surface: $\mathring{\vec{x}}_1, \ldots, \mathring{\vec{x}}_n \in \mathbb{R}^d$
- Move markers by ODE: $\begin{cases} \dot{\vec{x}}_k = \vec{v}(\vec{x}_k, t) \\ \vec{x}_k(0) = \mathring{\vec{x}}_k \end{cases}$

⊕ Fast, easy to move

⊕ Accurate (high order ODE solvers)

⊖ Uneven marker distribution



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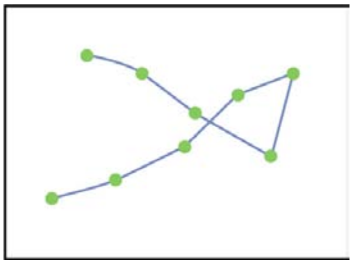
⊖ Incorrect entropy solution



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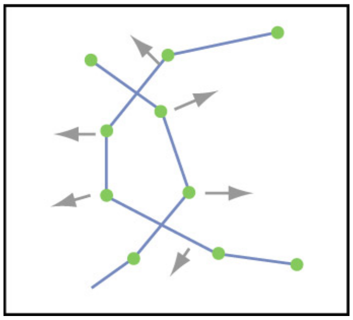
⊖ Topology changes



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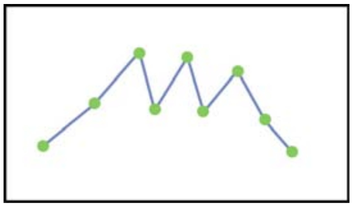
⊖ Numerical instabilities with curvature



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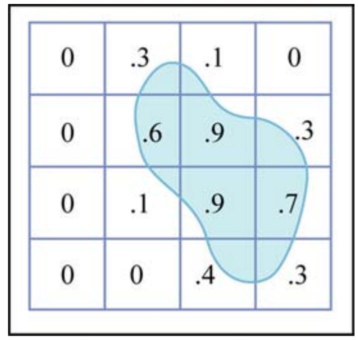
⊖ Marker connections in 3D?

II. Volume of Fluid:

- Regular grid
- Store volume/area "inside" surface
- Update volume value according to $\vec{v}$

$\oplus$ Very robust
$\oplus$ Simple in 3D
$\ominus$ Not very accurate
$\ominus$ Exact surface shape and topology?
$\ominus$ Curvature reconstruction?



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Implicit Representation

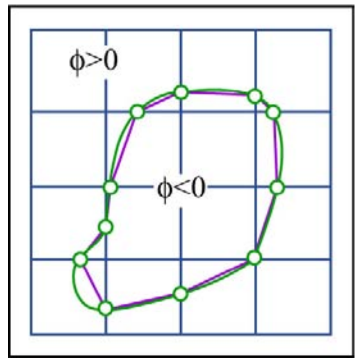
- Define function $\phi(\vec{x})$, s.t.

$$\left\{ \begin{array}{ll} \phi > 0 & \text{outside} \\ \phi = 0 & \text{interface} \\ \phi < 0 & \text{inside} \end{array} \right\}$$

- Store ϕ on regular Eulerian grid
- PDE IVP for ϕ, yielding correct movement
- Recover $\hat{n}, K$ from ϕ

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|}$$

$$\kappa = \nabla \cdot \left(\frac{\nabla \phi}{|\nabla \phi|} \right) = \frac{\phi_{xx} {\phi_y}^2 - 2\phi_x \phi_y \phi_{xy} + \phi_{yy} {\phi_x}^2}{({\phi_x}^2 + {\phi_y}^2)^{\frac{3}{2}}}$$



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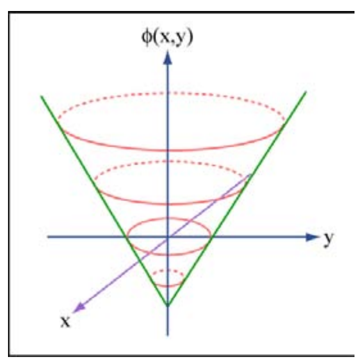
Approximate $\phi_x, \phi_y, \phi_{xx}, \phi_{xy}, \phi_{yy}$ by finite differences (e.g. central).

Signed Distance Function:

$$\left\{ \begin{array}{ll} \phi = 0 & \text{surface} \\ |\nabla \phi| = 1 & \text{almost everywhere} \\ \phi < 0 & \text{inside} \end{array} \right.$$

$\oplus$ Surface reconstruction very robust

$$\oplus \left\{ \begin{array}{l} \hat{n} = \nabla \phi \\ K = \nabla^2 \phi \end{array} \right.$$



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$\phi(x, y) = \sqrt{x^2 + y^2} - 1$
describes unit circle

PDE

Movement under given velocity field $\vec{v}$:
$\phi_t + \vec{v} \cdot \nabla \phi = 0$ Linear advection

Special case: normal velocity $\vec{v} = F\hat{n} = F\frac{\nabla \phi}{|\nabla \phi|}$

$\Rightarrow \boxed{\phi_t + F|\nabla \phi| = 0}$ Level set equation

Numerical Methods

- Upwind
- WENO
- (Spectral)

Ex.: Upwind for level set equation (first order)

$$\frac{\phi_{i,j}^{n+1}-\phi_{i,j}^{n}}{\Delta t} = \max(F,0)\nabla_{ij}^{+} + \min(F,0)\nabla_{ij}^{-}$$

$$\nabla_{ij}^{+} = (\max(D^{-x}\phi,0)^2 + \min(D^{+x}\phi,0)^2 + \max(D^{-y}\phi,0)^2 + \min(D^{+y}\phi,0)^2)^{\frac{1}{2}}$$

$$\nabla_{ij}^{-} = (\min(\underbrace{D^{-x}\phi}_{\text{all evaluated at } \phi_{i,j}^{n}},0)^2 + \max(D^{+x}\phi,0)^2 + \min(D^{-y}\phi,0)^2 + \max(D^{+y}\phi,0)^2)^{\frac{1}{2}}$$

Higher order: WENO and SSP-RK.

Reinitialization

Desirable $|\nabla\phi| = 1$.
But in general $\phi_t + F|\nabla\phi| = 0$ does not preserve $|\nabla\phi| = 1$.

Fixes:

- Solve IVP

$\phi_\tau + \text{sign}(\phi)(|\nabla\phi| - 1) = 0$
In each time step, for $0 \leq \tau \leq$?

- Solve Eikonal equation

Given ϕ, find $\hat{\phi}$, s.t.

$$\left\{ \begin{array}{l} |\nabla\hat{\phi}| = 1 \\ \{\hat{\phi} = 0\} = \{\phi = 0\} \end{array} \right\}$$

Use fast marching method by Sethian.

- Extension velocity:

Change velocity field $\vec{v}$ to $\hat{\vec{v}}$, s.t.

$$\left\{ \begin{array}{l} \hat{\vec{v}} = \vec{v} \text{ at } \{\phi = 0\} \\ \nabla\hat{\vec{v}} = 0 \rightarrow |\nabla\phi| = 1 \text{ preserved} \end{array} \right.$$