Navier-Stokes Equations

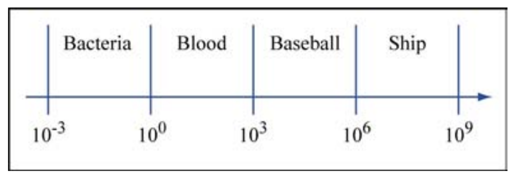
$$\begin{cases} \underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} = -\nabla p + \frac{1}{\mathrm{Re}} \nabla^2 \underline{u} \; [+g] & \text{Momentum equation} \\ \nabla \cdot \underline{u} = 0 & \text{Incompressibility} \end{cases}$$

Incompressible flow, i.e. density $\rho =$ constant.

Reynolds number:

$$\mathrm{Re} = \frac{U \cdot L}{\nu} = \frac{\text{inertial forces}}{\text{viscous forces}}$$

$U =$ Characteristic velocity
$L =$ Characteristic length scale
$\nu =$ Kinetic viscosity



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in 2D: $\underline{u} = \begin{pmatrix} u \\ v \end{pmatrix}$

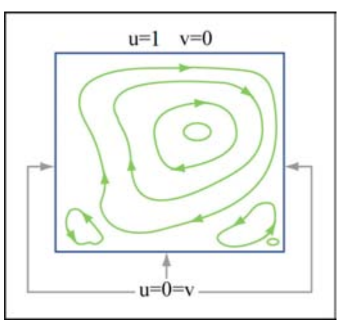
(1) $u_t + uu_x + vu_y = -p_x + \frac{1}{\mathrm{Re}}(u_{xx} + u_{yy})$

(2) $v_t + uv_x + vv_y = -p_y + \frac{1}{\mathrm{Re}}(v_{xx} + v_{yy})$

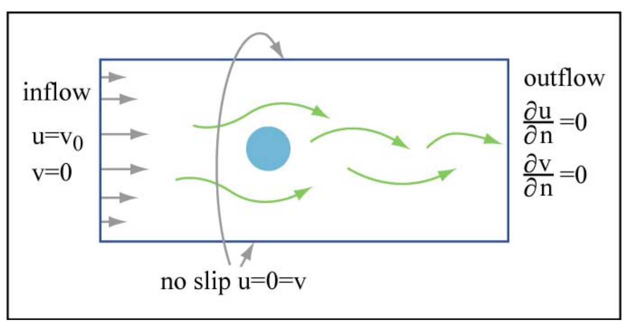
(3) $u_x + v_y = 0$

<u>Famous Problems</u>:

Lid driven cavity



Flow around cylinder



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3 unknowns, 3 equations

DAE (Differential Algebraic System), (3) is a constraint

Solve by projection approach:

In each time step

I. Solve $\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} = \frac{1}{\text{Re}} \nabla^2 \underline{u}$

$$\frac{U^* - U^n}{\Delta t} = -(U^n \cdot \nabla) U^n + \frac{1}{\text{Re}} \nabla^2 U^n$$

Note: $\nabla \cdot U^* \neq 0$

II. Project on divergence-free velocity field

$$\frac{U^{n+1} - U^*}{\Delta t} = -\nabla p$$

What is $p : 0 \overset{!}{=} \nabla \cdot U^{n+1} = \nabla \cdot U^* - \Delta t \nabla^2 P$

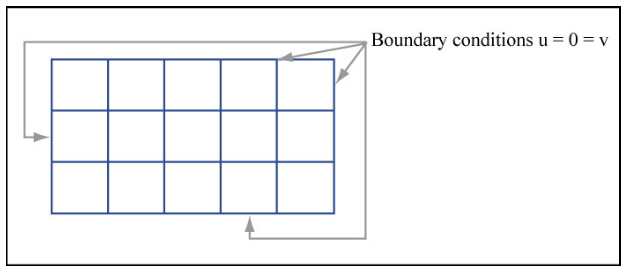
$\Rightarrow \nabla^2 p = \frac{1}{\Delta t} \nabla \cdot U^*$ Poisson equation for pressure

Discretization:

Solution:

$u = v = 0,$

$p = \text{constant}$



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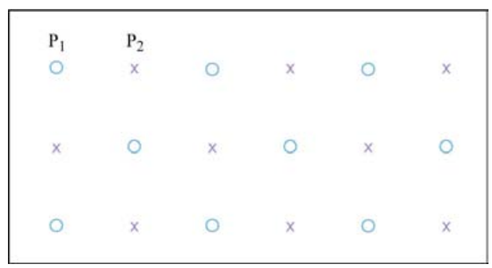
But: Central differences on grid allow solution

$U_{ij} = V_{ij} = 0,$

$$P_{ij} = \left\{ \begin{array}{ll} P_1 & \text{for} \quad i+j \text{ even} \\ P_2 & \quad\quad\;\; i+j \text{ odd} \end{array} \right\}$$



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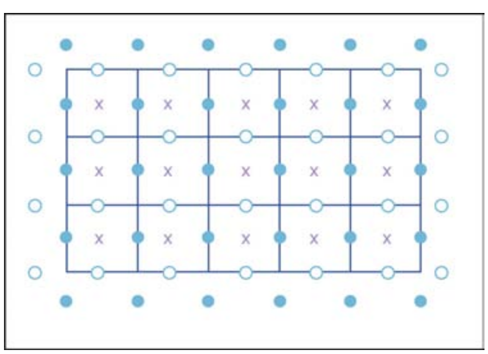
Fix: Staggered grid



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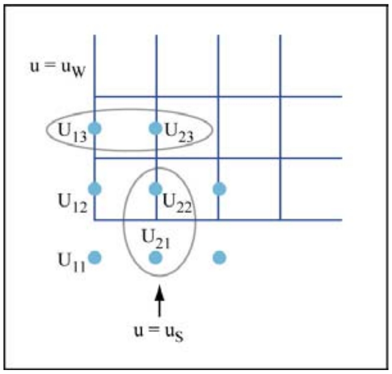
$\times$ pressure p

$\bullet$ velocity u

$\circ$ velocity v

Boundary Conditions:



$$U_{13} = u_w$$

$$\frac{U_{21} + U_{22}}{2} = u_s \Rightarrow U_{21} + U_{22} = 2u_s$$

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Numerical Method:

I. a) Treat Nonlinear Terms

$$uu_x + vu_y = (u^2)_x + (uv)_y$$
$$uv_x + vv_y = (uv)_x + (v^2)_y \qquad (\text{use } u_x + v_y = 0)$$

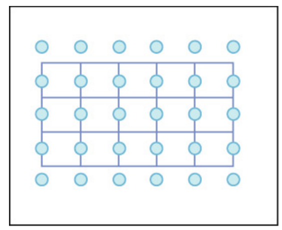
$$\left[\frac{\partial(U^2)}{\partial x}\right]_{ij} = \frac{(U_{i+\frac{1}{2},j})^2 - (U_{i-\frac{1}{2},j})^2}{\Delta x}$$

$$\left[\frac{\partial(UV)}{\partial y}\right]_{ij} = \frac{U_{i,j+\frac{1}{2}}V_{i,j+\frac{1}{2}} - U_{i,j-\frac{1}{2}}V_{i,j-\frac{1}{2}}}{\Delta y}$$

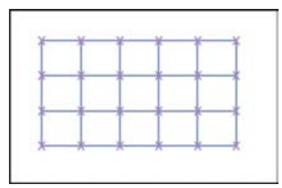
$$\left[\frac{\partial(UV)}{\partial x}\right]_{ij} = \frac{U_{i+\frac{1}{2},j}V_{i+\frac{1}{2},j} - U_{i-\frac{1}{2},j}V_{i-\frac{1}{2},j}}{\Delta x}$$

$$\left[\frac{\partial(V^2)}{\partial y}\right]_{ij} = \frac{(V_{i,j+\frac{1}{2}})^2 - (V_{i,j-\frac{1}{2}})^2}{\Delta y}$$

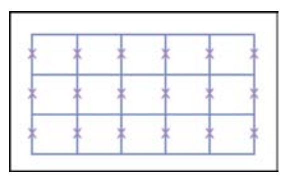
U_{ij}



$U_{i,j+\frac{1}{2}}$



$\left[\frac{\partial(UV)}{\partial y}\right]_{ij}$



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where $U_{i+\frac{1}{2},j} = \frac{U_{i,j} + U_{i+1,j}}{2}$, $U_{i,j+\frac{1}{2}} = \frac{U_{i,j} + U_{i,j+1}}{2}$

$$\frac{U^*_{i,j} - U^n_{i,j}}{\Delta t} = -\left[\frac{\partial(U^2)}{\partial x}\right]^n_{i,j} - \left[\frac{\partial(UV)}{\partial y}\right]^n_{i,j}$$

$$\frac{{V_{i,j}}^* - {V_{i,j}}^n}{\Delta t} = -\left[\frac{\partial(UV)}{\partial x}\right]^n_{i,j} - \left[\frac{\partial(V^2)}{\partial y}\right]^n_{i,j}$$

I. b) Implicit Diffusion

$$\frac{\underline{U}^{**} - \underline{U}^{*}}{\Delta t} = \frac{1}{\mathrm{Re}} K2D \cdot \underline{U}^{**}$$

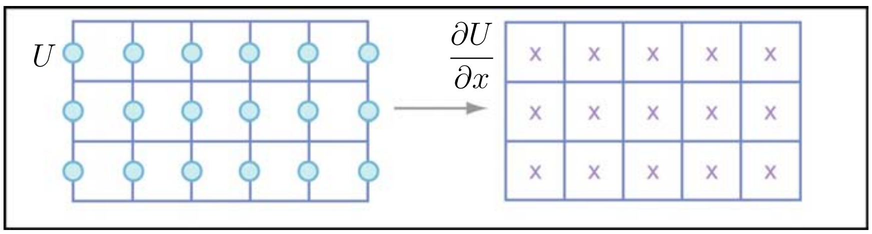
$$\frac{\underline{V}^{**} - \underline{V}^{*}}{\Delta t} = \frac{1}{\mathrm{Re}} \underset{\uparrow}{K2D} \cdot \underline{V}^{**}$$

5 point Laplace stencil with Dirichlet boundary conditions

II. Pressure Correction

$$K2D \cdot \underline{P} = \frac{1}{\Delta t}\left(\frac{\partial \underline{U}^{**}}{\partial x} + \frac{\partial \underline{V}^{**}}{\partial y}\right)$$

with Neumann boundary conditions $\frac{\partial p}{\partial n} = 0$



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$$\frac{\underline{U}^{n+1} - \underline{U}^{**}}{\Delta t} = -\frac{\partial \underline{P}}{\partial x}$$

$$\frac{\underline{V}^{n+1} - \underline{V}^{**}}{\Delta t} = -\frac{\partial \underline{P}}{\partial y}$$

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