Four Important Linear PDE

Laplace/Poisson equation

$$\begin{cases} -\nabla^2 u = f & \text{in } \Omega \\ u = g & \text{on } \Gamma_1 \leftarrow \text{Dirichlet boundary condition} \\ \dfrac{\partial u}{\partial n} = h & \text{on } \Gamma_2 \leftarrow \text{Neumann boundary condition} \end{cases}$$

$$\Gamma_1 \dot\cup \Gamma_2 = \partial\Omega$$

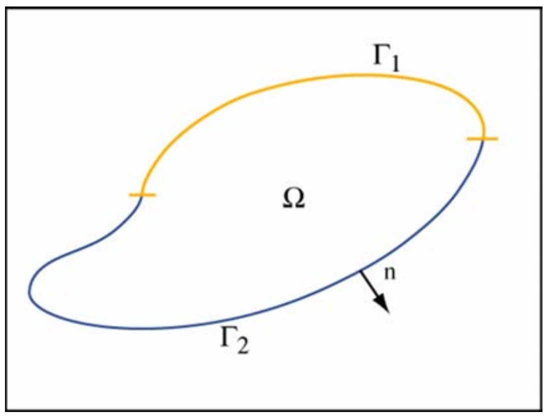

Image by MIT OpenCourseWare.

$f \equiv 0 \longrightarrow$ Laplace equation

$$\nabla^2 u = 0$$

$u =$ "harmonic function"

Physical example:

Heat equation: $u_t - \nabla^2 u = \underbrace{f}_{\text{source}}$

stationary $(t \to \infty) : u_t = 0 \Rightarrow -\nabla^2 u = f$

Dirichlet: prescribe $u = g$

Neumann: prescribe flux $\dfrac{\partial u}{\partial n} = h$

Fundamental solution of Laplace equation:

$$\begin{cases} \Omega = \mathbb{R}^n \\ \text{no boundary conditions} \end{cases}$$

Radially symmetric solution in $\mathbb{R}^n \setminus \{0\}$:

$$r = |x| = \left(\sum_{i=1}^{n} {x_i}^2 \right)^{\frac{1}{2}} \qquad \left[\begin{array}{l} \dfrac{\partial r}{\partial x_i} = \dfrac{2x_i}{2|x|} = \dfrac{x_i}{r} \\ \dfrac{\partial^2 r}{\partial {x_i}^2} = \dfrac{1 \cdot r - x_i \frac{\partial r}{\partial x_i}}{r^2} = \dfrac{1}{r} - \dfrac{{x_i}^2}{r^3} \end{array} \right]$$

$$u(x) = v(r)$$

$$\Rightarrow u_{x_i} = v'(r) \frac{\partial r}{\partial x_i}$$

$$\Rightarrow u_{x_i x_i} = v''(r) \left(\frac{\partial r}{\partial x_i} \right)^2 + v'(r) \frac{\partial^2 r}{\partial {x_i}^2} = v''(r) \frac{{x_i}^2}{r^2} + v'(r) \cdot \left(\frac{1}{r} - \frac{{x_i}^2}{r^3} \right)$$

$$\Rightarrow \nabla^2 u = \sum_{i=1}^{n} u_{x_i x_i} = v''(r) + v'(r) \cdot \frac{n-1}{r}$$

Hence:

$$\nabla^2 u = 0 \iff v''(r) + \frac{n-1}{r} v'(r) = 0$$

$$\overset{v' \neq 0}{\iff} (\log v'(r))' = \frac{v''(r)}{v'(r)} = \frac{1-n}{r}$$

$$\iff \log v'(r) = (1-n) \log r + \log b$$

$$\iff v'(r) = b \cdot r^{1-n}$$

$$\iff v(r) = \left\{ \begin{array}{ll} br + c & n = 1 \\ b \log r + c & n = 2 \\ \dfrac{b}{r^{n-2}} + c & n \geq 3 \end{array} \right\}$$

Def.: The function

$$\Phi(x) = \left\{ \begin{array}{ll} -\frac{1}{2}|x| & n = 1 \\ -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\alpha(n)} \cdot \frac{1}{|x|^{n-2}} & n \geq 3 \end{array} \right\} (x \neq 0, \alpha(n) = \text{volume of unit ball in } \mathbb{R}^n)$$

is called fundamental solution of the Laplace equation.

Rem.: In the sense of distributions, Φ is the solution to

$$-\nabla^2 \Phi(x) = \underbrace{\delta(x)}_{\text{Dirac delta}}$$

Poisson equation:

Given $f : \mathbb{R}^n \to \mathbb{R}$,

$u(x) = \displaystyle\int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy$ (convolution)

solves $-\nabla^2 u(x) = f(x)$.

Motivation:

$$\nabla^2 u(x) = \int_{\mathbb{R}^2} -\nabla_x{}^2 \Phi(x-y) f(y)\, dy = \int_{\mathbb{R}^2} \delta(x-y) f(y)\, dy = f(x).$$

Φ is a Green's function for the Poisson equation on $\mathbb{R}^n$.

Properties of harmonic functions:

Mean value property

$$u \text{ harmonic} \iff u(x) = \underset{\substack{\uparrow \\ \text{average}}}{⨍}_{\partial B(x,r)} u\, ds \iff u(x) = \underset{\substack{\uparrow \\ \text{average}}}{⨍}_{B(x,r)} u\, dy$$

for any ball $B(x,r) = \{y : ||y-x|| \le r\}$.

Implication: u harmonic $\Rightarrow u \in C^\infty$

Proof: $u(x) = \int_{\mathbb{R}^n} \chi_{B(0,r)}(x-y) u(y)\, dy$

$$u \in C^k \overset{\text{convolution}}{\Longrightarrow} u \in C^{k+1} \square$$

Maximum principle

Domain $\Omega \subset \mathbb{R}^n$ bounded.

(i) u harmonic $\Rightarrow \max\limits_{\overline{\Omega}} u = \max\limits_{\partial\Omega} u$ (weak MP)

(ii) Ω connected; u harmonic

If $\exists\, x_0 \in \Omega : u(x_0) = \max\limits_{\overline{\Omega}} u$, then $u \equiv$ constant (strong MP)

Implications

- $u \to -u \Rightarrow \max \to \min$
- uniqueness of solution of Poisson equation with Dirichlet boundary conditions

$$\left\{ \begin{array}{rl} -\nabla^2 u = f & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{array} \right\}$$

Proof: Let u_1, u_2 be two solutions.

Then $w = u_1 - u_2$ satisfies

$$\left\{ \begin{array}{rl} \nabla^2 w = 0 & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{array} \right\} \overset{\text{max principle}}{\Longrightarrow} w \equiv 0 \Rightarrow u_1 \equiv u_2 \square$$

Pure Neumann Boundary Condition:

$$\left\{ \begin{array}{rl} -\nabla^2 u = f & \text{in } \Omega \\ \dfrac{\partial u}{\partial h} = h & \text{on } \partial\Omega \end{array} \right\}$$

has infinitely many solutions ($u \to u + c$), if $-\int_\Omega f dx = -\int_{\partial\Omega} h\, dS$.
Otherwise no solution.

Compatibility Condition:

$$-\int_\Omega f\, dx = \int_\Omega \nabla^2 u\, dx = \int_\Omega \mathrm{div} \nabla f\, dx = \int_{\partial\Omega} \nabla f \cdot n\, dS = \int_{\partial\Omega} \frac{\partial f}{\partial n}\, dS = \int_{\partial\Omega} h\, dS.$$