Spectral Methods

[Reference: Trefethen, Spectral Methods in MATLAB, SIAM 2000]

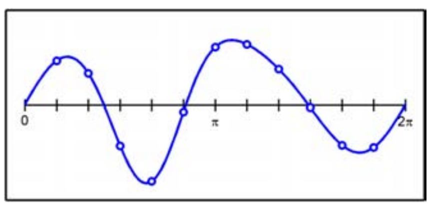
Classical Methods: error $= O(h^p)$, $\quad p = 1, 2, 3...$ fixed
Spectral: error $= O(h^p) \quad \forall\, p$
Error decays (with h) faster than any polynomial order
e.g. error $= O(h^{\frac{1}{h}})$ exponential decay
Only true if solution smooth: $u \in C^\infty$
Otherwise: $u \in C^p, u \notin C^{p+1} \Rightarrow$ error $= O(h^p)$

Message 1:

Spectral methods have a restricted area of application:
Linear problems on simple domains with simple boundary conditions and smooth solution. [often times subproblems]
But for those, they are awesome.

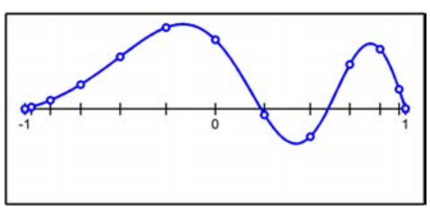
Two Cases:

1. Periodic: $\Omega = [0, 2\pi]$, where "$0 = 2\pi$"
   $u(x + 2\pi) = u(x)$



Use trigonometric functions: $u(x) = \sum_k c_k e^{ikx}$

2. Non-Periodic: $\Omega =]-1, 1[$



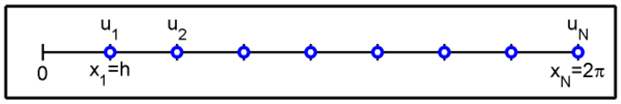
Use polynomials on Chebyshev points (non-equidistant).

Message 2:

FD/FE/FV methods are local.
Spectral methods are global.

Periodic Domains

$\Omega = [0, 2\pi]$, "$0 = 2\pi$", $u(x) = u(x + 2\pi)$
Uniform grid



Task: Approximate $u'(x_i) \approx \sum_j \alpha_{ij} u_j$

$$O(h): u'(x_i) \approx \frac{u_{i+1} - u_i}{h}$$

$$O(h^2): u'(x_i) \approx \frac{u_{i+1} - u_{i-1}}{2h} \qquad \text{(3 point)}$$

$$O(h^4): u'(x_i) \approx \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12h} \qquad \text{(5 point)}$$

$$O(h^6): u'(x_i) \approx \frac{u_{i+3} - 9u_{i+2} + 45u_{i+1} - 45u_{i-1} + 9u_{i-2} - u_{i-3}}{60h} \qquad \text{(7 point)}$$

$\downarrow$

Limit: use all points; expect $O(h^N) = O(h^{\frac{1}{h}})$

$$u'(x_i) \approx \frac{1}{2}\cot\left(\frac{h}{2}\right) \cdot (u_{i+1} - u_{i-1}) - \frac{1}{2}\cot\left(\frac{2h}{2}\right) \cdot (u_{i+2} - u_{i-2}) + \frac{1}{2}\cot\left(\frac{3h}{2}\right) \cdot (u_{i+3} - u_{i-3}) - \dots$$

Limit $N \to \infty$ "infinite grid"

$$u'(x_i) \approx \frac{1}{h} \cdot (u_{i+1} - u_{i-1}) - \frac{1}{2h} \cdot (u_{i+2} - u_{i-2}) + \frac{1}{3h} \cdot (u_{i+3} - u_{i-3}) - \dots$$

Matrix Notation:

$\vec{u} = (u_1, u_2, \dots, u_N)^T$
$\vec{w} = (w_1, w_2, \dots, w_N)^T \overset{!}{\approx} (u'(x_1), \dots, u'(x_N))^T$
$\vec{w} = D \cdot \vec{u}$

5 point stencil:

$$D = \begin{bmatrix} 0 & \frac{2}{3} & -\frac{1}{12} & & & \frac{1}{12} & -\frac{2}{3} \\ -\frac{2}{3} & \ddots & \ddots & \ddots & & & \frac{1}{12} \\ \frac{1}{12} & \ddots & \ddots & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \ddots & \ddots & -\frac{1}{12} \\ -\frac{1}{12} & & & \ddots & \ddots & \ddots & \frac{2}{3} \\ \frac{2}{3} & -\frac{1}{12} & & & \frac{1}{12} & -\frac{2}{3} & 0 \end{bmatrix} \quad \text{banded sparse matrix}$$

Spectral $N = 6$:

$$D_6 = \begin{bmatrix} 0 & \alpha_1 & -\alpha_2 & \alpha_3 & -\alpha_4 & \alpha_5 \\ -\alpha_1 & 0 & \alpha_1 & -\alpha_2 & \alpha_3 & -\alpha_4 \\ \alpha_2 & -\alpha_1 & 0 & \alpha_1 & -\alpha_2 & \alpha_3 \\ -\alpha_3 & \alpha_2 & -\alpha_1 & 0 & \alpha_1 & -\alpha_2 \\ \alpha_4 & -\alpha_3 & \alpha_2 & -\alpha_1 & 0 & \alpha_1 \\ -\alpha_5 & \alpha_4 & -\alpha_3 & \alpha_2 & -\alpha_1 & 0 \end{bmatrix} \quad \underline{\text{full}} \text{ matrix}$$

$$\alpha_j = \tfrac{1}{2}\cot\left(\tfrac{jh}{2}\right)$$

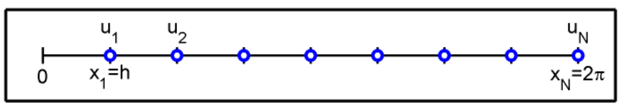
How to obtain spectral differentiation matrices?

<u>Fourier Basis:</u>

u periodic on $[0, 2\pi] = \Omega$

$\Rightarrow u(x) = \sum_{k\in\mathbb{Z}} \hat{u}_k e^{ikx}$ (Fourier series, Ω bounded)

u only known at grid points $x_j = jh$



where $h = \dfrac{2\pi}{N} \Rightarrow \boxed{\dfrac{\pi}{h} = \dfrac{N}{2}}$ [here: N even]

$$\Rightarrow u_j = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{u}_k e^{ikx_j} \quad \text{discrete Fourier series}$$

$$\begin{bmatrix} \text{physical space} & : & \text{discrete,} & \text{bounded} \\ & & \updownarrow & \updownarrow \\ \text{Fourier space} & : & \text{bounded,} & \text{discrete} \end{bmatrix}$$

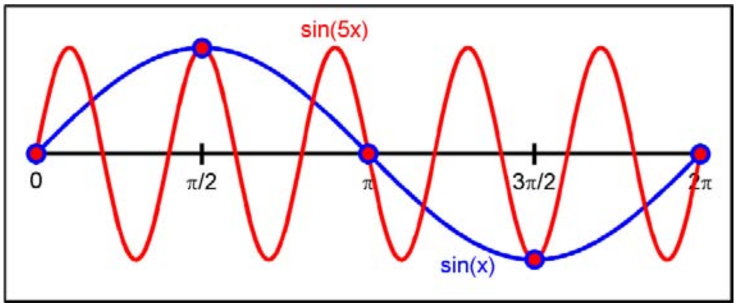
$$\hat{u}_k = h \sum_{j=1}^{N} u_j e^{-ikx_j} \ \forall k = -\frac{N}{2}+1, \ldots, \frac{N}{2}$$

Discrete Fourier Transform (DFT) $\longrightarrow$ Use FFT to do in $O(N \log N)$!

$$u_j = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \hat{u}_k e^{ikx_j} = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}}{}' \, \hat{u}_k e^{ikx_j}$$

where $\displaystyle\sum_{k=-\frac{N}{2}}^{\frac{N}{2}}{}' c_k = \tfrac{1}{2}c_{-\frac{N}{2}} + c_{-\frac{N}{2}+1} + \cdots + c_{\frac{N}{2}-1} + \tfrac{1}{2}c_{\frac{N}{2}}, \quad \hat{u}_{-\frac{N}{2}} = \hat{u}_{\frac{N}{2}}$

Finite Fourier sum due to grid aliasing:



$\sin(x) = \sin(5x)$ on the grid $x_j = 2\pi \frac{j}{N}$

Nyquist sampling theorem

At grid points:

$$u(x_j) = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}}{}' \hat{u}_k e^{ikx_j}$$

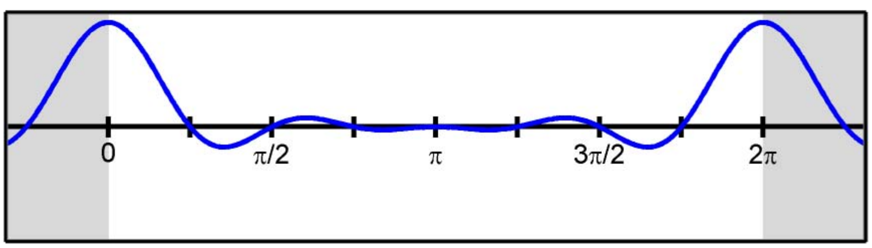
Define interpolant:

$$p(x) = \frac{1}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}}{}' \hat{u}_k e^{ikx} \qquad \text{Band-Limited Interpolant (BLI)}$$

Trigonometric polynomial of degree $\leq \frac{N}{2}$

Basis function:

BLI for $\delta_j \left\{ \begin{array}{ll} 1 & j = 0 \,(\text{mod } N) \\ 0 & j \neq 0 \,(\text{mod } N) \end{array} \right\}$



$$u_j = \delta_j \xrightarrow{\text{DFT}} \hat{u}_k = h \sum_{j=1}^{N} \delta_j e^{-ikx_j} = h \;\forall k$$

$$\Rightarrow p(x) = \frac{h}{2\pi} \sum_{k=-\frac{N}{2}}^{\frac{N}{2}}{}' e^{ikx} = \cdots = \frac{\sin(\frac{\pi x}{h})}{\frac{2\pi}{h}\tan(\frac{x}{2})} =: S_N(x)$$

"periodic sinc function"

Arbitrary grid values:

$(u_1, u_2, \ldots, u_N)$

$$u_j = \sum_{m=1}^{N} u_m \delta_{j-m} \Rightarrow p(x) = \sum_{m=1}^{N} u_m S_N(x - x_m)$$

Differentiation:

Differentiate BLI $p(x) \longrightarrow p'(x)$

$$S'_N(x_j) = \left\{ \begin{array}{ll} 0 & j = 0\,(\mathrm{mod}\ N) \\ \frac{1}{2}(-1)^j \cot(\frac{jh}{2}) & j \neq 0\,(\mathrm{mod}\ N) \end{array} \right\}$$

$$\Rightarrow D_N = \left[\begin{array}{cccccc} \ddots & \ddots & \ddots & \ddots & & \\ \ldots & -\frac{1}{2}\cot(\frac{h}{2}) & 0 & \frac{1}{2}\cot(\frac{h}{2}) & -\frac{1}{2}\cot(\frac{2h}{2}) & \ldots \\ & & \ddots & \ddots & \ddots & \ddots \end{array} \right]$$

$$D_N{}^{(2)} = \left[\begin{array}{ccc} \ddots & \vdots & \\ \ddots & -\frac{1}{2}\csc(\frac{2h}{2}) & \\ \ddots & \frac{1}{2}\csc^2(\frac{h}{2}) & \ddots \\ \ddots & -\frac{\pi^2}{3h} - \frac{1}{6} & \ddots \\ \ddots & \frac{1}{2}\csc^2(\frac{h}{2}) & \ddots \\ & -\frac{1}{2}\csc^2(\frac{2h}{2}) & \ddots \\ & \vdots & \ddots \end{array} \right] = D_N{}^2$$

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