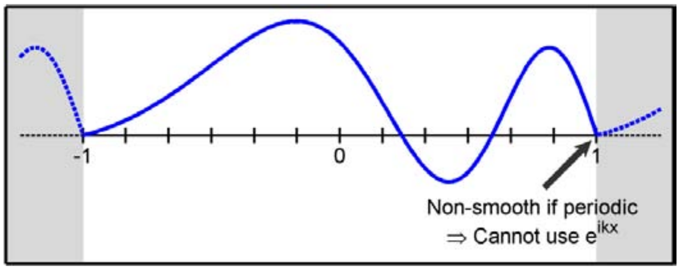
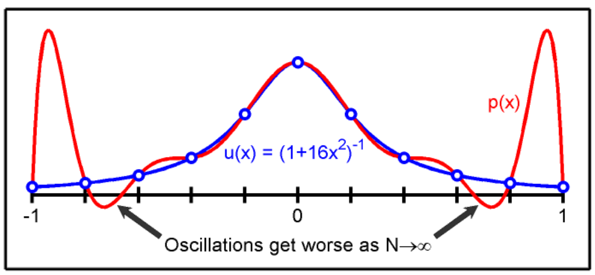
Non-periodic Domains



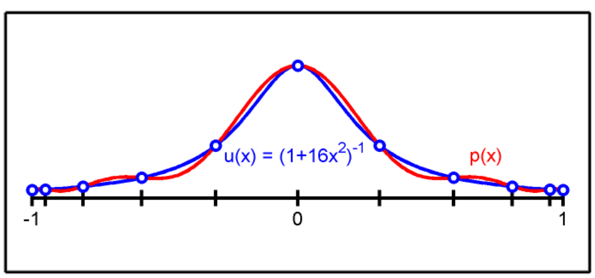
So use algebraic polynomials $p(x) = a_0 + a_1 x + \cdots + a_N x^N$

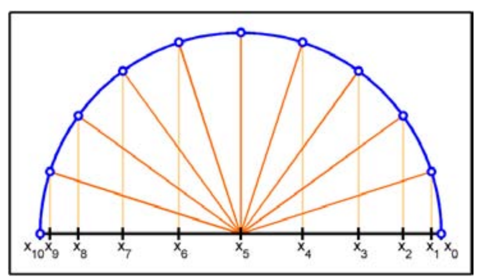
Problem: Runge phenomenon on equidistant grids



$p(x) \not\to u(x)$ as $N \to \infty$

Remedy: Chebyshev points $\qquad x_j = \cos\left(\pi \frac{j}{N}\right)$





$P_N(x) \to u(x)$ as $N \to \infty$

Spectral Differentiation:

Given $(u_0, u_1, \ldots, u_N) \rightsquigarrow$ interpolating polynomial

$$p(x) = \sum_{j=0}^{N} u_j \frac{\prod_{k\neq j}(x - x_k)}{\prod_{k\neq j}(x_j - x_k)}$$

$$\downarrow$$

$$p'(x) \to p'(x_j) \to D_{ij} = \left\{ \begin{array}{cc} \dfrac{a_i}{a_j(x_i - x_j)} & i \neq j \\ \displaystyle\sum_{k\neq j} \frac{1}{x_j - x_k} & i = j \end{array} \right\}$$

$$a_j = \prod_{k\neq j}(x_j - x_k)$$

Chebyshev Differentiation Matrix:

$$D_N = \begin{array}{|c|c|c|} \hline \dfrac{2N^2+1}{6} & 2\dfrac{(-1)^j}{1-x_j} & \frac{1}{2}(-1)^N \\ \hline -\frac{1}{2}\dfrac{(-1)^i}{1-x_i} & \begin{array}{c} \hfill \dfrac{(-1)^{i+j}}{x_i - x_j} \\ \dfrac{-x_j}{2(1-x_j^2)} \\ \dfrac{(-1)^{i+j}}{x_i - x_j} \hfill \end{array} & \frac{1}{2}\dfrac{(-1)^{N+i}}{1+x_i} \\ \hline -\frac{1}{2}(-1)^N & -2\dfrac{(-1)^{N+j}}{1+x_j} & -\dfrac{2N^2+1}{6} \\ \hline \end{array}$$

where $x_j = \cos(\frac{j\pi}{N}), j = 0, \ldots, N$

$\vec{w} = D_N \cdot u(\vec{x}) \approx u'(\vec{x})$ with spectral accuracy.
$\vec{w} = {D_N}^2 \cdot u(\vec{x}) \approx u''(\vec{x})$ with spectral accuracy.
etc.

Chebyshev Differentiation using FFT:

1. Given $u_0, \ldots, u_N$ at $x_j = \cos(\frac{j\pi}{N})$.
   Extend: $\vec{U} = (u_0, u_1, \ldots, u_N, u_{N-1}, \ldots, u_1)$
2. FFT: $\hat{U}_k = \frac{\pi}{N} \sum_{j=1}^{2N} e^{-ik\theta_j} U_j, \ k = -N+1, \ldots, N$
3. $\hat{W}_k = ik\hat{U}_k, \hat{W}_N = 0$ (first derivative)
4. Inverse FFT: $W_j = \frac{1}{2\pi} \sum_{k=-N+1}^{N} e^{ik\theta_j} \hat{W}_k, \ j = 1, \ldots, 2N$
5. $$\begin{cases} w_j = -\dfrac{W_j}{\sqrt{1 - {x_j}^2}}, & j = 1, \ldots, N-1 \\ w_0 = \dfrac{1}{2\pi} {\sum_{n=0}^{N}}' n^2 \hat{u}_n, & w_N = \dfrac{1}{2\pi} {\sum_{n=0}^{N}}' (-1)^{n+1} n^2 \hat{u}_n \end{cases}$$

$$p(x) = P(\theta), \ x = \cos\theta$$

$$p(x) = \sum_{n=0}^{N} \alpha_n T_n(x)$$

$$P(\theta) = \sum_{n=0}^{N} \alpha_n \cos(n\theta)$$

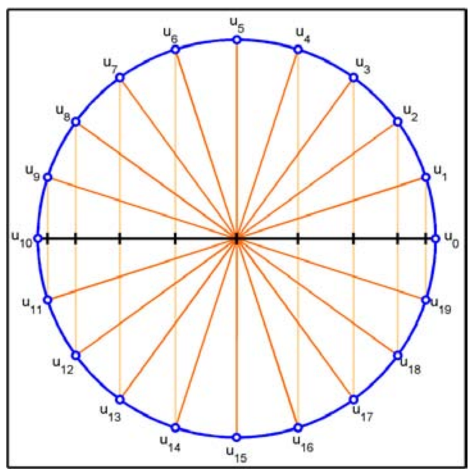
$T_n(x) =$ Chebyshev polynomial

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

$$p'(x) = \frac{P'(\theta)}{\frac{dx}{d\theta}} = \frac{-\sum_{n=0}^{N} n\alpha_n \sin(n\theta)}{-\sin\theta}$$

$$= \frac{\sum_{n=0}^{N} n\alpha_n \sin(n\theta)}{\sqrt{1-x^2}}$$

Visualization for $N = 10$:



Boundary value problems

Ex.: $\left\{\begin{array}{l} u_{xx} = e^{4x},\ x \in]-1,1[\\ u(\pm 1) = 0 \end{array}\right\}$ Poisson equation with homogeneous Dirichlet boundary conditions

Chebyshev differentiation matrix D_N.

Remove boundary points:

$$\left[\begin{array}{c} w_0 \\ \hline w_1 \\ \vdots \\ w_{N-1} \\ \hline w_N \end{array}\right] = \underbrace{\left[\begin{array}{c|c|c} & & \\ \hline & \tilde{D}_N^2 & \\ \hline & & \end{array}\right]}_{D_N^2} \cdot \left[\begin{array}{c} v_0[=0] \\ \hline v_1 \\ \vdots \\ v_{N-1} \\ \hline v_N[=0] \end{array}\right] \Bigg\} \text{Interior points} \qquad \boxed{\text{p13.m}}$$

Linear system:

$\tilde{D}_n^2 \cdot \vec{u} = \vec{f}$

where $\vec{u} = (u_1, \ldots, u_{N-1})$, $\vec{f} = (e^{4x_1}, \ldots, e^{4x_{N-1}})$

Nonlinear Problem

Ex.: $\left\{\begin{array}{l} u_{xx} = e^{u},\ x \in]-1,1[\\ u(\pm 1) = 0 \end{array}\right\}$ $\boxed{\text{p14.m}}$

initial guess ↙

Need to iterate: $\vec{u}^{(0)} = 0, \vec{u}^{(1)}, \vec{u}^{(2)}, \ldots$

$\tilde{D}_N^2 \vec{u}^{(k+1)} = \exp(\vec{u}^{(k)})$ ← fixed point iteration

Can also use Newton iteration...

Eigenvalue Problem

Ex.: $\left\{\begin{array}{l} u_{xx} = \lambda u,\ x \in]-1,1[\\ u(\pm 1) = 0 \end{array}\right\}$ $\boxed{\text{p15.m}}$

Find eigenvalues and eigenvectors of matrix $\tilde{D}_N^2$

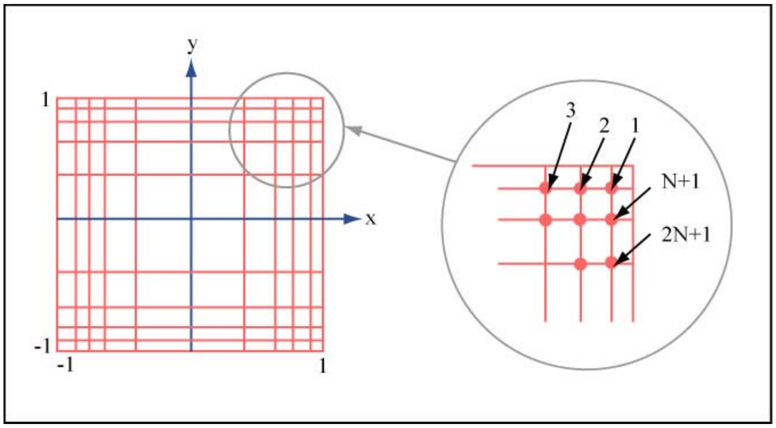
Matlab:

```
>> [V,L] = eig(D2)
```

Higher Space Dimensions

Ex.: $\left\{\begin{array}{ll} u_{xx} + u_{yy} = f(x,y) & \Omega =]-1,1[^2 \\ u = 0 & \partial\Omega \end{array}\right\}$ $f(x,y) = 10\sin(8x(y-1))$

Tensor product grid: $(x_i, y_j) = (\cos(\frac{i\pi}{N}), \cos(\frac{j\pi}{N}))$



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p16.m

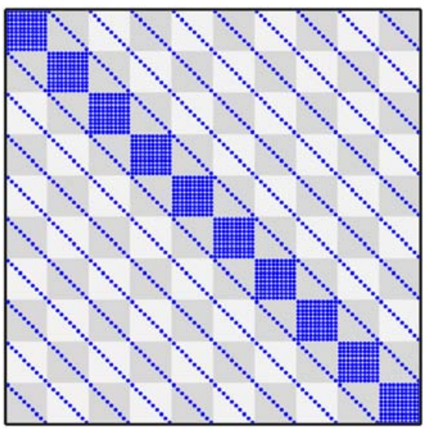
Matrix approach:

Matlab: `kron`, $\otimes$

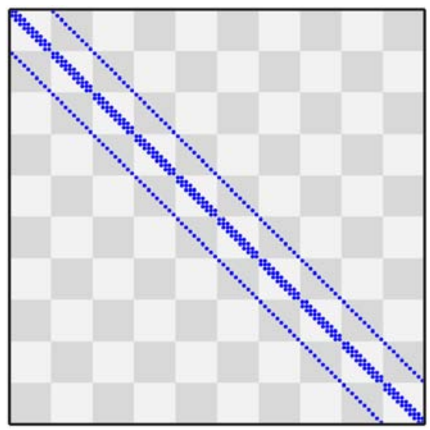
$L_N = I \otimes \tilde{D}_N^2 + \tilde{D}_N^2 \otimes I$

```
>> L = kron(I,D20)+kron(d2,I)
```

Linear system: $L_N \cdot \vec{u} = \vec{f}$



spectral



classical FD

<u>Helmholtz Equation</u>

Wave equation with source

$-v_{tt} + v_{xx} + v_{yy} = e^{ikt} f(x,y)$

Ansatz: $v(x,y,t) = e^{ikt} u(x,y)$

$\Rightarrow$ Helmholtz equation:

$$\begin{cases} u_{xx} + u_{yy} + k^2 u = f(x,y) & \Omega =]-1,1[^2 \\ u = 0 & \partial\Omega \end{cases}$$

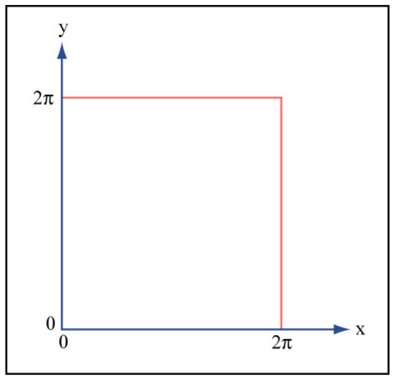
p17.m

Fourier Methods

So far spectral on grids (pseudospectral).
Can also work with Fourier coefficients directly.

Ex.: Poisson equation

$$\begin{cases} -(u_{xx}+u_{yy}) = f(x,y) & \Omega = [0,2\pi]^2 \\ \text{periodic boundary conditions} & \end{cases}$$



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$$f(x,y) = \sum_{k,l\in\mathbb{Z}} \hat{f}_{kl} e^{i(kx+ly)}$$

$$u(x,y) = \sum_{k,l\in\mathbb{Z}} \hat{u}_{kl} e^{i(kx+ly)}$$

$$u_{xx}(x,y) = \sum_{k,l} \hat{u}_{kl} e^{i(kx+ly)} \cdot (-k^2)$$

$$-\nabla^2 u(x,y) = \sum_{k,l} \hat{u}_{kl}(k^2+l^2) e^{i(kx+ly)} \stackrel{!}{=} f(x,y)$$

$$\Rightarrow \boxed{\hat{u}_{kl} = \frac{\hat{f}_{kl}}{k^2+l^2} \; \forall\, (k,l) \neq (0,0)}$$

$\hat{u}_{00}$ arbitrary constant, condition on $f : \hat{f}_{00} = 0$

In Fourier basis, differential operators are diagonal.