

Problem Set 6

**Due at 12:01 pm on Friday, October 19th, in the box provided.** No late psets are accepted. If you collaborated with other students in the class, list their names on the title sheet. The work that you submit must be your own.

**Main concepts: Limit cycles. Van der Pol oscillator.**

**Reading: Strogatz Ch. 7**

**Problem 1: Limit cycles**

1. Sketch the phase portrait in the  $(x, y)$ -plane for the system given in polar coordinates  $\dot{\theta} = r^2 - 2, \dot{r} = r(1 - r^2)(r + 3)$ .
2. Sketch the phase portrait in the  $(x, y)$ -plane for the system  $\dot{\theta} = 1, \dot{r} = \mu r + r^3$ , for  $\mu = -1, -0.36, -0.01, 0, 0.3$ . Describe what happens when  $\mu$  passes through 0.

**Problem 2: Numerical study of the Van der Pol oscillator.**

Consider an electric circuit shown in Figure 1 with the input-output relation for the tunnel diode given by:

$$i = \phi_t(v) = \phi(v - E_0) + I_0$$

with  $\phi(v) = \gamma v^3 - \alpha v$ . Here  $i$  is current, and  $v$  is voltage. The equation for the circuit can be written as:

$$\begin{aligned} \dot{V} &= \frac{1}{C}(-\phi(V) - W), \\ \dot{W} &= \frac{1}{L}V \end{aligned} \tag{1}$$

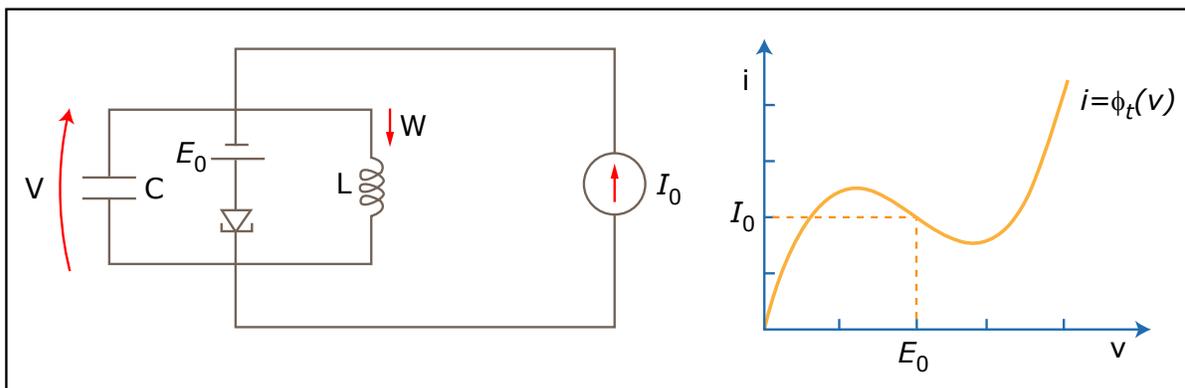


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Figure 1: An electrical circuit with a tunnel diode for the van der Pol oscillator.

1. Show that the two above equations can be rewritten as:

$$\ddot{V} - \frac{1}{C}(\alpha - 3\gamma V^2)\dot{V} + \frac{1}{LC}V = 0.$$

Non-dimensionalize it and bring it to the standard form of the Van der Pol equation:

$$\ddot{x} + \varepsilon(x^2 - 1)\dot{x} + x = 0.$$

2. Show that for small  $\varepsilon \ll 1$  the amplitude of the limit cycle is about 2. What is the period of the limit cycle? Does the period depend on the initial conditions? Contrast this behavior to the limit case  $\varepsilon = 0$ . Document your answers with graphs of both the phase space and the displacement.
3. For large  $\varepsilon \gg 1$  the oscillations become relaxation oscillations. Show (numerically) that in this case the amplitude of the limit cycle is still equal to 2. To keep the values of the velocity bounded, choose  $\varepsilon$  less than or about equal to 10. Again, document your answer with graphs of both the phase space and the displacement. Does the period depend on the initial conditions? For very large  $\varepsilon \gg 1$  one can find an approximate period of the Van der Pol oscillator (see Strogatz, p. 212-215). Estimate the period of the limit cycle from your computed graphs. How does this period compare with the period derived for large  $\varepsilon \gg 1$  in Strogatz?

### Problem 3: Ruling out the limit cycles

Show that there are no limit cycles in the system using Index theory, or (Tue lecture) Dulac's criterion, Liapunov function or the fact that the system is a gradient system.

1.  $\dot{x} = x + y^2$ ,  $\dot{y} = \sin x + 4y$
2.  $\dot{x} = x(1 - y)$ ,  $\dot{y} = y(1 - x)$
3.  $\dot{x} = y - x^3$ ,  $\dot{y} = -x - y^5$   
(Hint: try a Liapunov function of the form  $V(x, y) = ax^2 + by^2$ , with suitable  $a$  and  $b$ .)
4.  $\dot{x} = 2xy + \cos(y)$ ,  $\dot{y} = x^2 - x \sin(y)$

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